Dawson College: Winter 2020: Linear Algebra (Science): 201-NYC-05-S3: **Test 2** name: Y. Lamontagne

No books, watches, notes or cell phones are allowed. The **only** calculators allowed are the Sharp EL-531. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

**Question 1.** *(1 mark per blank)* [1] Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. If $A$ is a product of elementary matrices, then $\det(A)$ __cannot__ equal zero.

b. Let A be a $3 \times 3$ matrix, ~~and let $B$ be a $4 \times 4$ matrix. If the leading ones of the RREF of $A$ is equal to those~~ of the RREF of $B$, then $\det(A)$ __might__ equal zero and $\det(B)$ __must__ equal zero.

**Question 2.**[1] *(1 mark per blank)* Let $A$ and $B$ be invertible $n \times n$ matrices. Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. $A + B$ __might__ be invertible.

b. $A^T A$ __must__ be invertible.

**Question 3.** *(1 mark per blank)* Given $A$ an $n \times n$ matrix and $k$ a non-zero scalar.

a. If $A$ is an elementary matrix obtained by interchanging two rows then $\det(A) =$ __$-1$__.

b. If $A$ is the reduced row echelon form of an invertible matrix then $\det(A) =$ __$1$__.

c. If $A$ is a singular matrix then $\det(A) =$ __$0$__.

d. If $A$ is an elementary matrix obtained by adding $k$ times one row to an other then $\det(A) =$ __$1$__.

e. If $A$ is an elementary matrix obtained by multiplying one row by $k$ then $\det(A) =$ __$K$__.

f. If $A$ is the identity matrix multiplied by $k$ then $\det(A) =$ __$K^n$__.

**Question 4.**[1] ~~(5 marks)~~ Suppose $A$ is an $n \times n$ matrix such that $\det(A) = 5$ and $\det(2A^T) = 40$.

a. *(3 marks)* Find $n$.

b. *(2 marks)* Find $\det(\text{adj}(3A))$.

a)
$$2^n \det(A^T) = 40$$
$$2^n \det(A) = 40$$
$$2^n (5) = 40$$
$$2^n = 8 (= 2^3)$$
$$n = 3$$

b)
$$= (\det(3A))^{n-1}$$
$$= (3^n \det(A))^{n-1}$$
$$= (3^3\, 5)^{3-1}$$
$$= (3^3)^2 5^2$$
$$= 3^6 5^2$$

**Question 5.**[1] *(5 marks)* Given that $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = 10$ and $A = \begin{bmatrix} 3g+a & 3h+b & 2 & 3i+c \\ d+2a & e+2b & 3 & f+2c \\ a & b & 4 & c \\ 0 & 0 & 5 & 0 \end{bmatrix}$

a. *(5 marks)* Find $\det(A)$.

b. *(3 marks)* Using Cramer's Rule find $x_1$ and $x_3$ for $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = [2a \quad 3a \quad 4a \quad 5a]^T$

a)
$$|A| = \underbrace{a_{41}C_{41}}_{0} + \underbrace{a_{42}C_{42}}_{0} + a_{43}C_{43} + \underbrace{a_{44}C_{44}}_{0}$$
$$= 5(-1)^{4+3} \begin{vmatrix} 3g+a & 3h+b & 3i+c \\ d+2a & e+2b & f+2c \\ a & b & c \end{vmatrix}$$
$$= -5 \begin{vmatrix} 3g & 3h & 3i \\ d & e & f \\ a & b & c \end{vmatrix} \text{ after } \begin{matrix} -R_3+R_1 \to R_1 \\ -2R_3+R_2 \to R_2 \end{matrix}$$
$$= -5(3) \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} \text{ after } \tfrac{1}{3}R_1 \to R_1$$
$$= -5(3)(-1) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 15(10) = 150$$

b)
$$|A_1| = \begin{vmatrix} 2a & 3h+b & 2 & 3i+c \\ 3a & e+2b & 3 & f+2c \\ 4a & b & 4 & c \\ 5a & 0 & 5 & 0 \end{vmatrix} = 0 \text{ since } C_1 = aC_3$$
$$\therefore x_1 = \frac{|A_1|}{|A|} = \frac{0}{150} = 0$$
$$|A_3| = \begin{vmatrix} 3g+a & 3h+b & 2a & 3i+c \\ d+2a & e+2b & 3a & f+2c \\ a & b & 4a & c \\ 0 & 0 & 5a & 0 \end{vmatrix}$$
$$= a|A| \text{ after } \tfrac{1}{a}R_3 \to R_3 \text{, note if } a = 0 \text{ then } |A_3| = 0 \text{ and } x_3 = 0$$
$$= a150$$
$$\therefore x_3 = \frac{|A_3|}{|A|} = \frac{a150}{150} = a$$

**Question 6.** *(2 marks)* Prove: If the entries in each row of an $n \times n$ matrix $A$ add up to zero, then the determinant of $A$ is zero.

Let $\underline{x} = [1\ 1 \cdots 1]^T$, then $A\underline{x} = \underline{0}$ hence a non-trivial solution. So by the equivalence thm, $\det(A) = 0$.

[1] From or modified from a John Abbott final examination

**Question 7.** Given $A^{-1} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 0 & 0 & 1 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$.

a. *(5 marks)* Find $\det(A)$.

b. *(2 marks)* Find $\operatorname{adj}(A)$.

$$|A^{-1}| = \underbrace{a_{12}C_{12}}_{0} + \underbrace{a_{22}C_{22}}_{0} + a_{32}C_{32} + \underbrace{a_{42}C_{42}}_{0}$$

$$= a_{32}C_{32}$$

$$= -2(-1)^{3+2} \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 0 & 5 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 5 & 1 \end{vmatrix} \text{ after } R_1 + R_2 \to R_2$$

$$= 2\left[a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}\right]$$

$$= 2(1)(-1)^{1+1} \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = 2(-18) = -36$$

$$\det(A) = \frac{1}{\det(A^{-1})} = \frac{1}{-36}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$\operatorname{adj}(A) = \det(A)\, A^{-1}$$

$$= \frac{1}{-36} \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 0 & 0 & 1 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$$

**Question 8.**[1] *(3 marks each)* Determine whether the following statements are true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. If $A$ is a symmetric $n \times n$ matrix where $n$ is even then $\det(A) = 0$.

b. If $A$ is a skew-symmetrix $n \times n$ matrix where $n$ is odd then $\det(A) = 0$.

a) False,

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is symmetric

since $I_2^T = I_2$, $n = 2$ (even)

but $\det(I_2) = 1$.

b) True,

Since $A$ is skew-symmetric

$$A^T = -A$$

$$\det(A^T) = \det(-A)$$

$$\det(A) = (-1)^n \det(A)$$

$\det(A) = -\det(A)$ since $n$ is odd

$\therefore \det(A) = 0$

**Bonus Question.** *(2 marks)* Find $\det(A)$ where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

given that the numbers are $\mathbb{Z}_3$ instead of $\mathbb{R}$. Operations on the numbers of $\mathbb{Z}_3$ can be defined by the following Cayley tables:

| + | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 2 | 0 |
| 2 | 2 | 0 | 1 |

| × | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |