Dawson College: Winter 2022: Calculus II (SCIENCE): 201-NYB-05-S1: **Test 1, part 1 of 3** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show and justify all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Formulae:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. Calculus III must be extremely fun.

b. Suppose $f(x)$ be a continuous odd function on the interval $[-2, 2]$ then $\int_{-1}^{2} f(x)\,dx$ might be equal to zero.

c. The mean value theorem for integrals states that if $f(x)$ is continuous on $[a, b]$ then there must exists a number c in $[a, b]$ such that $\int_a^b f(x)\,dx = f(c)(b - a)$.

d. $\int_a^b f(x)\,dx$ might be equal to $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ where $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$.

e. $\int_a^b f(x)\,dx$ must be equal to $\int_a^b f(\alpha)\,d\alpha$.

Question 2. *(2 marks)* Suppose $a_m, a_{m+1}, \ldots, a_n$ and k are real numbers, prove $\sum_{i=m}^{n} ka_i = k \sum_{i=m}^{n} a_i$.

$$LHS = \sum_{i=m}^{n} ka_i$$
$$= ka_m + ka_{m+1} + \ldots + ka_n$$
$$= k(a_m + a_{m+1} + \ldots + a_n)$$
$$= k \sum_{i=m}^{n} a_i = RHS$$

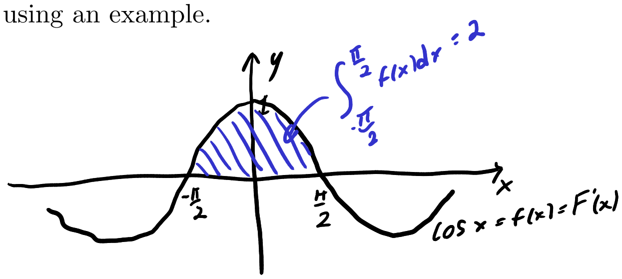
Question 3a. *(2 marks)* State the entire Fundamental Theorem of Calculus (FTC).

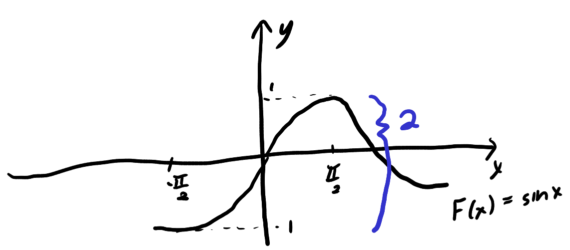
If $f(x)$ is a continuous function then

1. $\int_a^b f(x)dx = F(b) - F(a)$ where $F(x)$ is an antiderivative.

2. $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$ where $a \in \mathbb{R}$

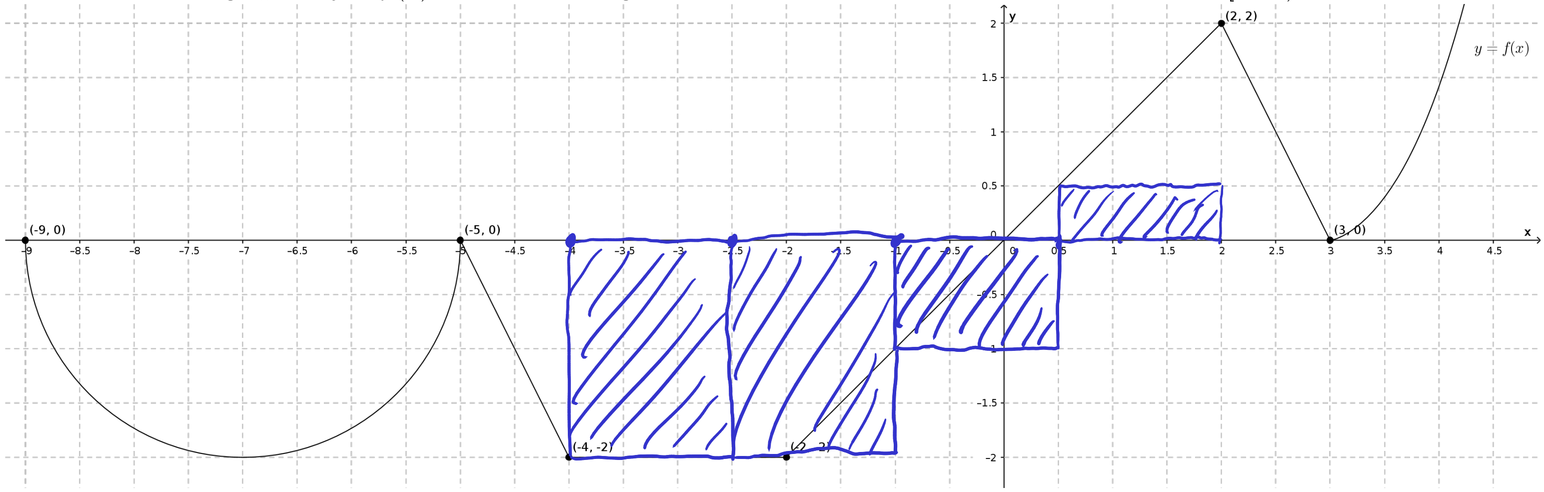
Question 3b. *(2 marks)* As seen in class the *Net Change Theorem* is an application of a part of the FTC. Explain the Net Change Theorem using an example.





The net change in y for $F(x)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ is equal to $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F'(x)dx$.

Question 3. The graph of $y = f(x)$ consists of straight lines, one semicircle and a curve on the interval $[3, \infty)$.



a. *(5 marks)* Find an approximation of the definite integral of $f(x)$ on the interval $[-4, 2]$, using the left endpoint as sample points and 4 approximating rectangles. Draw the approximating rectangles. Is the approximation an overestimate or underestimate? Justify.

b. *(2 marks)* Evaluate $\int_{-4}^{2} f(x)\,dx$.

c. *(4 marks)* Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n} + f\left(-4 + \frac{6i}{n}\right)\right) \frac{6}{n}$.

a) $\Delta x = \frac{b-a}{n} = \frac{2-(-4)}{4} = 1.5$

$x_i = -4 + i\Delta x = -4 + i(1.5)$

$x_0 = -4$

$x_1 = -2.5$

$x_2 = -1$

$x_3 = -0.5$

$\int_{-4}^{2} f(x)dx \approx f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$

$= (-2)1.5 + (-2)1.5 + (-1)1.5 + 0.5(1.5) = -6.75$

The approximation is an underestimate since the function is increasing on $[-4, 2]$ which implies $f(x_i) \le f(x) \le f(x_{i+1})$ $\forall x \in [x_i, x_{i+1}]$ and since always using left endpoint it guarantees an under estimate.

b) $\int_{-4}^{2} f(x)dx = -2(3) - \frac{2(1)}{2} + \frac{2(2)}{2} = -4$

c) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n} + f\left(-4 + \frac{6i}{n}\right)\right)\frac{6}{n}$

$= \lim_{n\to\infty} \left[\frac{6}{n^2}\sum_{i=1}^{n} i + \sum_{i=1}^{n} f\left(-4 + \frac{6i}{n}\right)\frac{6}{n}\right]$

$= \lim_{n\to\infty} \left[\frac{6}{\cancel{n^2}} \cdot \frac{\cancel{n}(n+1)}{2} + \sum_{i=1}^{n} f\left(-4 + \frac{6i}{n}\right)\frac{6}{n}\right]$

$= \lim_{n\to\infty} \left[\frac{3(n+1)}{n} + \text{"}\right]$

$= 3 + \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-4 + \frac{6i}{n}\right)\frac{6}{n}$

$= 3 + \int_{-4}^{2} f(x)dx$

since $\Delta x = \frac{b-a}{n} = \frac{6}{n}$

$x_i = a + i\Delta x$

$= -4 + i\frac{6}{n}$

where $a = -4$

which implies

$b = 2$

$= 3 - 4$

$= -1$

Question 4. Given the function

$$f(x) = \frac{1}{2} \int_e^{x^2} \frac{\arctan(\ln t)}{t}\, dt.$$

a. *(5 marks)* Evaluate $f(x)$.

b. *(5 marks)* Determine whether $f(x)$ is a solution to the initial value problem (IVP) given below

$$y' = \arctan(2\ln x), \quad y\left(\sqrt{e}\right) = 0.$$

a) $f(x) = \frac{1}{2}\int_{\frac{\pi}{4}}^{x^2} \frac{\arctan(\ln t)}{t}\,dt$

$z = \ln t$

$dz = \frac{1}{t}dt$

$z(x^2) = \ln x^2$

$z(e) = \ln(e) = 1$

$= \frac{1}{2}\int_1^{\ln x^2} \arctan z\, dz$

$= \frac{1}{2}\left[[uv]_1^{\ln x^2} - \int_1^{\ln x^2} v\, du \right]$

$u = \arctan z \qquad du = \frac{1}{1+z^2}dz$

$v = z \qquad dv = dz$

$= \frac{1}{2}[z \arctan z]_1^{\ln x^2} - \frac{1}{2}\int_1^{\ln x^2} \frac{z}{1+z^2}\,dz$

$= \frac{1}{2}[z\arctan z]_1^{\ln x^2} - \frac{1}{2}\left[\frac{1}{2}\ln|1+z^2|\right]_1^{\ln x^2}$

$= \frac{1}{2}\ln x^2 \arctan(\ln x^2) - \frac{1}{2}\arctan 1 - \frac{1}{4}\ln(1+(\ln x^2)^2) + \frac{1}{4}\ln(1+1^2)$

$= \frac{1}{4}\ln 2 - \frac{\pi}{8} + \frac{1}{2}\ln x^2 \arctan(\ln x^2) - \frac{1}{4}\ln 2\,(1+(\ln x^2)^2)$

b) Does $f(x)$ satisfy the differential equation?

$y' = f'(x) = \frac{d}{dx}[h(g(x))]$

$= h'(g(x))g'(x)$

where $h(x) = \frac{1}{2}\int_e^x \frac{\arctan(\ln t)}{t}dt$

and $h'(x) = \frac{1}{2}\frac{\arctan x}{x}$ by the 2^{nd} FTC

and $g(x) = x^2$

$g'(x) = 2x$

$\therefore\ y' = \frac{1}{\cancel{2}}\frac{\arctan(\ln x^2)}{x^{\cancel{2}}} \cdot \cancel{2}x$

$= \frac{\arctan(2\ln x)}{x}$

Does it satisfy the initial condition

$y(\sqrt{e}) = \int_e^{(\sqrt{e})^2} \frac{\arctan(\ln t)}{t}dt$

$= \int_e^e \frac{\arctan(\ln t)}{t}dt$

$= 0$

$\therefore$ $f(x)$ is a solution to the IVP since it satisfies the differential equation and initial condition.

Question 5. *(5 marks)* Find the average of the function $f(x) = |\tan^3(x)\sec^3(x)|$ on $[-\pi/6, \pi/6]$.

avg. value of function on $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

$$= \frac{1}{\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)} \int_{-\pi/6}^{\pi/6} |\tan^3(x)\sec^3(x)|\,dx$$

$$= \frac{6}{2\pi} \cdot 2 \int_0^{\pi/6} |\tan^3 x \sec^3 x|\,dx \quad \text{since } f(x) \text{ is even}$$

$f(x)$ is an even func since

$f(-x) = |\tan^3(-x)\sec^3(-x)|$
$= |(-\tan x)^3 \sec^3 x|$

since $\tan x$ is odd and $\sec^2 x$ is even

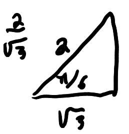
$= |\tan^3 x \sec^3 x|$
$= f(x)$

$$= \frac{6}{\pi} \int_0^{\pi/6} \tan^3 x \sec^3 x\,dx \quad \text{since } \tan x, \sec x \geq 0 \text{ on } \left[0, \frac{\pi}{6}\right]$$

$$= \frac{6}{\pi} \int_0^{\pi/6} \tan^2 x \sec^2 x \tan x \sec x\,dx$$

$$= \frac{6}{\pi} \int_0^{\pi/6} (\sec^2 - 1)\sec^2 x \tan x \sec x\,dx$$

$u = \sec x$
$du = \sec x \tan x\,dx$
$u(\pi/6) = \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$
$u(0) = \sec 0 = 1$



$$= \frac{6}{\pi} \int_1^{2/\sqrt{3}} (u^2 - 1)u^2\,du$$

$$= \frac{6}{\pi} \int_1^{2/\sqrt{3}} (u^4 - u^2)\,du$$

$$= \frac{6}{\pi} \left[\frac{u^5}{5} - \frac{u^3}{3}\right]_1^{2/\sqrt{3}}$$

$$= \frac{6}{\pi}\left[\left[\frac{(2/\sqrt{3})^5}{5} - \frac{(2/\sqrt{3})^3}{3}\right] - \left[\frac{1^5}{5} - \frac{1^5}{3}\right]\right]$$

$$= \frac{6}{\pi}\left(\frac{2}{15} - \frac{8}{45\sqrt{3}}\right)$$

Question 6. [1] *(5 mark each)* Evaluate the following integrals:

a.

$$\int_0^{-\ln(2)} e^x\sqrt{2e^x - e^{2x}}\,dx \quad \int_1^{1/2}\sqrt{2u-u^2}\,du$$

Let $u = e^x$

$du = e^x dx$

$u(1) = e^{\ln(\frac{1}{2})} = \frac{1}{2}$

$u(0) = e^0 = 1$

$$= \int_1^{1/2}\sqrt{1-(u-1)^2}\,du$$

$-u^2+2u$

$= -[u^2-2u]$

$= -[u^2-2u+1-1]$

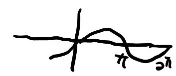
$= -[(u-1)^2-1]$

$= 1-(u-1)^2$

$$= \int_0^{-\frac{\pi}{6}} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$u-1 = \sin\theta$

$du = \cos\theta\,d\theta$



$$= \int_0^{-\frac{\pi}{6}} \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$\frac{1}{2}-1 = \sin\theta \Rightarrow \theta = -\pi/6$

$1-1 = \sin\theta \Rightarrow \theta = 0$

$$= \int_0^{-\frac{\pi}{6}} |\cos\theta|\cos\theta\,d\theta$$

$$= \int_0^{-\pi/6} \cos^2\theta\,d\theta \quad \text{since } \cos\theta \geq 0 \text{ on } [-\tfrac{\pi}{2}, -\tfrac{\pi}{6}]$$

$$= \int_0^{-\pi/6} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{-\frac{\pi}{6}} 1+\cos 2\theta\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{-\pi/6}$$

$$= \frac{1}{2}\left[\left[-\frac{\pi}{6} + \frac{\sin 2(-\pi/6)}{2}\right] - \left[0 + \overbrace{\frac{\sin 2(0)}{2}}^{0}\right]\right]$$

$$= -\frac{\pi}{12} - \frac{1}{2}\,\frac{\sqrt{3}/2}{2}$$

$$= -\frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

[1]From or modified from a John Abbott final examination

b.

$$\int \frac{x^3+x^2+x+2}{(x+1)(x^2+1)}\,dx$$

Note the above rational function is improper.

$(x+1)(x^2+1) = x^3+x+x^2+1 = x^3+x^2+x+1$

So

$$\int \frac{x^3+x^2+x+1+1}{x^3+x^2+x+1}\,dx = \int \cancel{\frac{x^3+x^2+x+1}{x^3+x^2+x+1}}\,dx + \int \frac{1}{x^3+x^2+x+1}\,dx$$

$$= \int 1\,dx + \int \frac{1}{(x+1)(x^2+1)}\,dx$$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$\frac{1\,\cancel{(x+1)}\cancel{(x^2+1)}}{\cancel{(x+1)}\cancel{(x^2+1)}} = \frac{A\cancel{(x+1)}(x^2+1)}{\cancel{x+1}} + \frac{(Bx+C)(x+1)\cancel{(x^2+1)}}{\cancel{x^2+1}}$$

$$1 = A(x^2+1) + (Bx+C)(x+1)$$

Let $x=-1$

$1 = A((-1)^2+1) + (B(-1)+C)(-1+1)$

$\frac{1}{2} = A$

Let $x=0$

$1 = A(0^2+1) + (B(0)+C)(0+1)$

$1 = \frac{1}{2} + C$

$\frac{1}{2} = C$

Let $x=1$

$1 = A((1)^2+1) + (B(1)+C)(1+1)$

$1 = \frac{1}{2}\cdot 2 + (B(1)+\frac{1}{2})2$

$\frac{-1}{2} = B$

$$= \int 1\,dx + \int \frac{1/2}{x+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}\,dx$$

$$= x + \frac{1}{2}\ln|x+1| - \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= x + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\arctan x + C$$

Bonus Question. *(5 marks)* Use the following integral

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$$

to show that $\pi < \frac{22}{7}$.