Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show and justify all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT. Suppose f and g are continuous functions such that $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty g(x)\,dx$ is convergent, then $\int_a^\infty f(x)\,dx$ might be convergent.
2. If $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ might be convergent.

Question 1. *(5 marks)* Evaluate the improper integral or show it diverges.

$$\int_0^1 \frac{\text{arcsin}x}{\sqrt{1-x^2}}\,dx$$

Infinite discontinuity at $x=0$

$$= \lim_{b\to 1^-} \int_0^b \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1^-} \int_0^{\arcsin b} u\,du$$

$u = \arcsin x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

$u(b) = \arcsin b$

$u(0) = \arcsin(0) = 0$

$$= \lim_{b\to 1^-} \left[\frac{1}{2}u^2\right]_0^{\arcsin b}$$

$$= \lim_{b\to 1^-} \left[\frac{1}{2}(\underbrace{\arcsin b}_{\to \pi/2})^2 - \frac{1}{2}(0)^2\right]$$

$$= \frac{\pi^2}{8}$$

Question 2. *(5 marks)* Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx$$

Let $g(x) = \frac{x}{x^2+1} - \frac{C}{3x+1}$

converges. Evaluate the integral for this value of C.

$$= \lim_{b \to \infty} \int_0^b \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx$$

$$= \lim_{b \to \infty} \left[\frac{1}{2} \ln(x^2+1) - \frac{C}{3} \ln|3x+1| \right]_0^b$$

$$= \lim_{b \to \infty} \left[\left[\frac{1}{2} \ln(b^2+1) - \frac{C}{3} \ln|3b+1| \right] - \left[\frac{1}{2} \overbrace{\ln(0^2+1)}^{0} - \frac{C}{3} \overbrace{\ln|3(0)+1|}^{0} \right] \right]$$

I.f. $\infty - \infty$, also $|3b+1| = 3b+1$ since $b \to \infty$

$$= \lim_{b \to \infty} \left[\ln(b^2+1)^{1/2} - \ln(3b+1)^{C/3} \right]$$

$$= \lim_{b \to \infty} \ln \frac{(b^2+1)^{1/2}}{(3b+1)^{C/3}}$$

$$= \ln \lim_{b \to \infty} \frac{(b^2+1)^{1/2}}{(3b+1)^{C/3}} \quad \text{since } \ln \text{ is continuous}$$

Lets look at $\lim_{b \to \infty} \frac{(b^2+1)^{1/2}}{(3b+1)^{C/3}} = \lim_{b \to \infty} \frac{\left(b^2\left(1+\frac{1}{b^2}\right)\right)^{1/2}}{\left(b\left(3+\frac{1}{b}\right)\right)^{C/3}}$

$$= \lim_{b \to \infty} \frac{b\left(1+\cancel{\frac{1}{b^2}}\right)^{1/2}}{b^{C/3}\left(3+\cancel{\frac{1}{b}}\right)^{C/3}}$$

Let $f(b) = \frac{b\left(1+\frac{1}{b^2}\right)}{b^{C/3}\left(3+\frac{1}{b}\right)^{C/3}}$

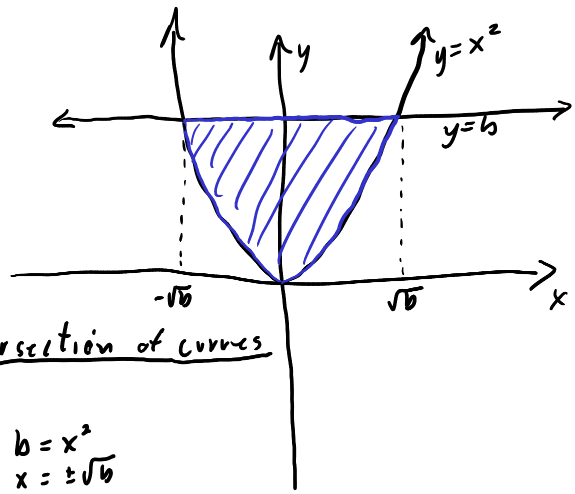
Case 1: $C = 3$ then $\lim_{b \to \infty} f(b) = \frac{1}{3}$ and $\int_0^\infty g(x) = \ln\left(\frac{1}{3}\right)$

Case 2: $C > 3$ then $\lim_{b \to \infty} f(b) = 0$ and $\ln(f(b)) \to -\infty$ as $b \to \infty$. $\therefore \int_0^\infty g(x)\,dx$ diverges

Case 3: $C < 3$ then $\lim_{b \to \infty} f(b)$ diverges to ∞ and $\ln(f(b)) \to \infty$ as $b \to \infty$ $\therefore \int_0^\infty g(x)\,dx$ diverges

Question 3. *(5 marks)* Sketch the region(s) enclosed by the given curves and find b such that the total area of the enclosed region(s) is 2022.

$y = x^2, \quad y = b, \quad \text{where } b > 0.$



Intersection of curves

$b = x^2$
$x = \pm\sqrt{b}$

$$2022 = \int_{-\sqrt{b}}^{\sqrt{b}} b - x^2\,dx$$

Let $f(x) = b - x^2$,
$f(-x) = b - (-x)^2$
$= b - x^2$
$= f(x)$
$\therefore f(x)$ is even

$\therefore$

$$2022 = 2\int_0^{\sqrt{b}} b - x^2\,dx$$

$$2022 = 2\left[bx - \frac{x^3}{3}\right]_0^{\sqrt{b}}$$

$$1011 = b\sqrt{b} - \frac{(\sqrt{b})^3}{3} - \left[b(0) - \frac{0^3}{3}\right]$$

$$1011 = b^{3/2} - \frac{b^{3/2}}{3}$$

$$1011 = \frac{2}{3}b^{3/2}$$

$$\frac{3}{2}(1011) = b^{3/2}$$

$$b = \left(\frac{3}{2}(1011)\right)^{2/3}$$

Question 4. For each of the following parts, set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis using the specified method. Sketch the region, sketch the solid,, draw a representative rectangle, write a representative element and label the sketch completely.

$x = (y-3)^2 + 1,\ x = 5$; about $y = 1$

Intersection of the two curves:

$5 = (y-3)^2 + 1$

$4 = (y-3)^2$

$\pm 2 = y - 3$

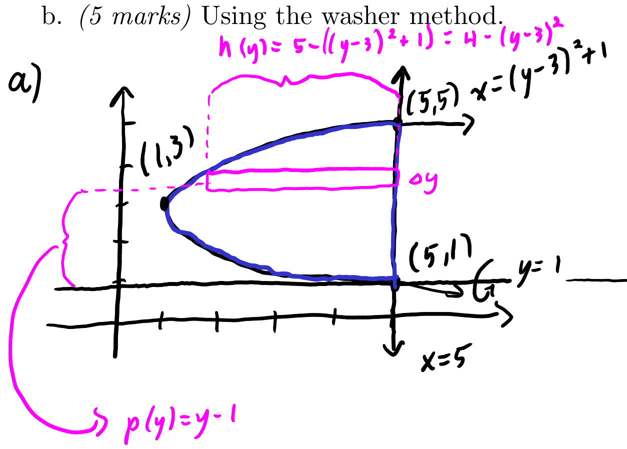
$y = 3 \pm 2$

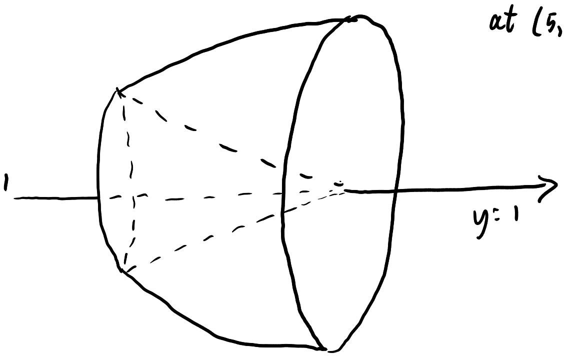
at $(5,1)$ and $(5,5)$

a. *(5 marks)* Using the shell method.

b. *(5 marks)* Using the washer method.

a)

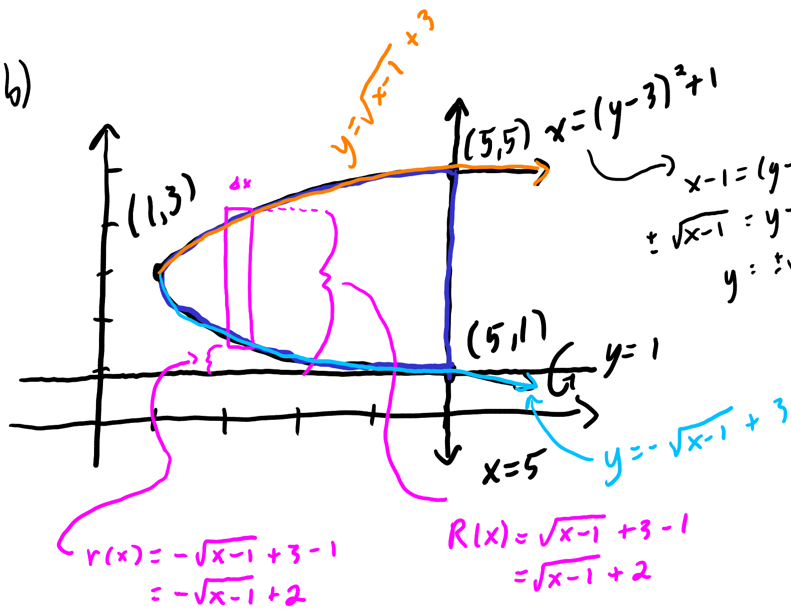




$$\Delta V = 2\pi\, p(y)\, h(y)\, \Delta y$$
$$= 2\pi (y-1)(4 - (y-3)^2)\, \Delta y$$

$$V = \int_1^5 2\pi (y-1)(4-(y-3)^2)\, dy$$

b)



$x - 1 = (y-3)^2$

$\pm\sqrt{x-1} = y - 3$

$y = \pm\sqrt{x-1} + 3$

$$\Delta V = \pi \left[(R(x))^2 - (r(x))^2 \right] \Delta x$$
$$= \pi \left[(\sqrt{x-1} + 2)^2 - (-\sqrt{x-1} + 2)^2 \right] \Delta x$$

$$V = \int_1^5 \pi \left[(\sqrt{x-1} + 2)^2 - (-\sqrt{x-1} + 2)^2 \right] dx$$

Question 5. [1] *(5 mark)* Find the length of the curve $y = \arcsin(x) + \sqrt{1-x^2}$ on its domain.

The domain of the curve is $[-1, 1]$

$$y' = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot -2x$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$(y')^2 = \frac{1}{1-x^2} - \frac{2x}{1-x^2} + \frac{x^2}{1-x^2}$$

$$= \frac{1-2x+x^2}{1-x^2}$$

$$1 + (y')^2 = 1 + \frac{1-2x+x^2}{1-x^2}$$

$$= \frac{1-x^2+1-2x+x^2}{1-x^2}$$

$$= \frac{2-2x}{1-x^2}$$

$$= \frac{2(1-x)}{(1-x)(1+x)}$$ removable discontinuity at $x=1$

$$= \frac{2}{1+x}$$

infinite discontinuity at $x=-1$

$$s = \int_{-1}^{1} \sqrt{1+(y')^2}\, dx$$

$$= \int_{-1}^{1} \sqrt{\frac{2}{1+x}}\, dx$$

$$= \sqrt{2} \int_{-1}^{1} \frac{1}{\sqrt{1+x}}\, dx$$

$$= \sqrt{2} \lim_{a \to -1^+} \int_{a}^{1} \frac{1}{\sqrt{1+x}}\, dx$$

$$= \sqrt{2} \lim_{a \to -1^+} \Big[2\sqrt{1+x} \Big]_a^1$$

$$= \sqrt{2} \lim_{a \to -1^+} \Big[2\sqrt{1+1} - 2\underbrace{\sqrt{1+a}}_{\to 0} \Big]$$

$$= \sqrt{2}\, 2\sqrt{2}$$

$$= 4$$

[1]From a John Abbott final examination

Question 6. *(5 marks)* Find the limit.

$$\lim_{x\to\infty} x \int_0^{1/x} \arctan(1-t)\ dt$$ indeterminate form $\infty \cdot 0$

$$= \lim_{x\to\infty} \frac{\int_0^{1/x} \arctan(1-t)dt}{1/x}$$ indeterminate form $\frac{0}{0}$

$$\overset{\hat{H}}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}\left[\int_0^{1/x} \arctan(1-t)dt\right]}{\frac{d}{dx}\left[1/x\right]}$$

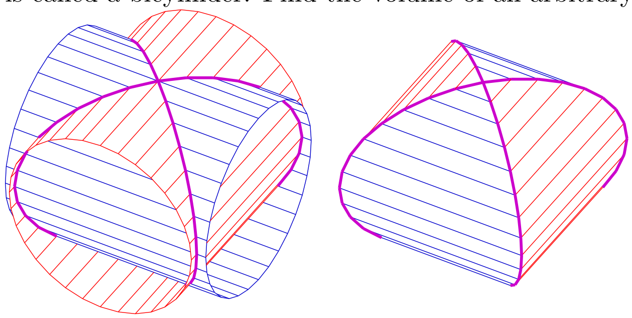
$$= \lim_{x\to\infty} \frac{\arctan(1-1/x)\cancel{(-1/x^2)}}{\cancel{-1/x^2}}$$ by the 2^{nd} FTC

$$= \lim_{x\to\infty} \arctan(1-\underbrace{1/x}_{\to 0})$$

$$= \arctan(1)$$

$$= \pi/4$$

Bonus Question.[2] *(5 marks)* In geometry, a Steinmetz solid is the solid body obtained as the intersection of two or three cylinders of equal radius at right angles. Each of the curves of the intersection of two cylinders is an ellipse. The intersection of two cylinders is called a bicylinder. Find the volume of an arbitrary bicylinder.



[2]https://en.wikipedia.org/wiki/Steinmetz_solid