Dawson College: Winter 2022: Calculus II (SCIENCE): 201-NYB-05-S1: **Test 3, part 1 of 3** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show and justify all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

**Question 1.** *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. If $a_n$ and $b_n$ are divergent, then $a_n + b_n$ might be divergent.

b. If $a_n > 0$ and $\lim_{n\to\infty}(a_{n+1}/a_n) < 1$ then $\lim_{n\to\infty} a_n$ must be equal to zero.

c. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ might be convergent.

d. A bounded sequence might converge.

e. If $\sum c_n x^n$ diverges when $x = 2$, then it might diverge when $x = 1$.

**Question 2.** *(5 marks)* Find the sum of the following series:

$$\sum_{n=2}^{\infty} [\operatorname{arcsec}(n) - \operatorname{arcsec}(n+1)]$$

Let $S_n$ be the $n^{th}$ partial sum.

$$S_n = a_2 + a_3 + a_4 + \dots + a_{n-2} + a_{n-1} + a_n$$
$$= [\operatorname{arcsec} 2 - \cancel{\operatorname{arcsec} 3}] + [\cancel{\operatorname{arcsec} 3} - \cancel{\operatorname{arcsec} 4}] + [\cancel{\operatorname{arcsec} 4} - \cancel{\operatorname{arcsec} 5}]$$
$$+ \dots + [\cancel{\operatorname{arcsec}(n-2)} - \cancel{\operatorname{arcsec}(n-1)}] + [\cancel{\operatorname{arcsec}(n-1)} - \cancel{\operatorname{arcsec}(n)}] + [\cancel{\operatorname{arcsec}(n)} - \operatorname{arcsec}(n+1)]$$
$$= \operatorname{arcsec} 2 - \operatorname{arcsec}(n+1)$$

$$S = \lim_{n\to\infty} S_n$$
$$= \lim_{n\to\infty} [\operatorname{arcsec} 2 - \underbrace{\operatorname{arcsec}(n+1)}_{\to \frac{\pi}{2}}]$$
$$= \operatorname{arcsec} 2 - \frac{\pi}{2}$$
$$= \frac{\pi}{3} - \frac{\pi}{2}$$
$$= \frac{-\pi}{6}$$

**Question 3.**[1] *(3 marks)* Let $\sum_{n=1}^{\infty} a_n$ be a series whose $n^{th}$ partial sum is given by $S_n = \frac{2n+1}{n+2}$. Find $a_9$.

① $S_9 = a_1 + a_2 + \dots + a_8 + a_9$

② $S_8 = a_1 + a_2 + \dots + a_8$

① - ②

$$S_9 - S_8 = a_9$$
$$a_9 = S_9 - S_8$$
$$= \frac{2(9)+1}{9+2} - \frac{2(8)+1}{8+2}$$
$$= \frac{19}{11} - \frac{17}{10}$$
$$= \frac{19(10) - 17(11)}{110} = \frac{3}{110}$$

[1]Modified from a John Abbott final examination

**Question 4.**[2]

a. *(1 mark)* Show that $0 < \frac{(n!)^2}{(2n)!} < 1$ when $n \geq 1$.

b. *(3 marks)* Use part (a) (whether or not you have shown it) to determine if the following sequence converges or diverges. If it converges, find its limit.

$$a_n = \frac{(n!)^2}{(2n+1)!}, \qquad n \geq 1.$$

a) Since the numerator and denominator can be written as a product of natural numbers, it is bounded below by 0.

$$0 < \frac{(n!)^2}{(2n)!} = \frac{n!\,n!}{(2n)!}$$

$$= \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{(n-1)}\cancel{n}\; n!}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{(n-1)}\cancel{n}(n+1)(n+2)\cdots(2n-1)(2n)}$$

$$= \frac{1\cdot 2\cdots(n-1)n}{(n+1)(n+2)\cdots(2n-1)2n}$$

$$= \underbrace{\frac{1}{n+1}}_{<1}\cdot\underbrace{\frac{2}{n+2}}_{<1}\cdots\underbrace{\frac{n-1}{2n-1}}_{<1}\cdot\underbrace{\frac{n}{2n}}_{<1}$$

$$< 1$$

b) From a) we have that

$$0 < \frac{(n!)^2}{(2n)!} < 1$$

$$\frac{0}{2n+1} < \frac{(n!)^2}{(2n)!\,(2n+1)} < \frac{1}{2n+1}$$

$$0 < \frac{(n!)^2}{(2n+1)!} < \frac{1}{2n+1}$$

Let $b_n = 0$ and $c_n = \frac{1}{2n+1}$, it follows that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$.

Then by the squeeze theorem, $\lim_{n\to\infty} a_n = 0$.

[2]From a Dawson College final examination

**Question 5.** *(5 marks)* Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{4^n \sqrt[5]{n+3}}$$

Lets determine the radius of convergence

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{(x+2)^{n+1}}{4^{n+1}\sqrt[5]{n+1+3}}}{\frac{(x+2)^n}{4^n\sqrt[5]{n+3}}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x+2)^{\cancel{n+1}}}{4^{n+1}\sqrt[5]{n+4}} \cdot \frac{\cancel{4^n}\sqrt[5]{n+3}}{\cancel{(x+2)^n}} \right|$$

$$= \lim_{n\to\infty} \frac{\sqrt[5]{n+3}}{4\sqrt[5]{n+4}} |x+2|$$

$$= \lim_{n\to\infty} \frac{1}{4}\sqrt[5]{\frac{n+3}{n+4}} |x+2|$$

$$= \lim_{n\to\infty} \frac{1}{4}\sqrt[5]{\frac{1+3/n}{1+4/n}} |x+2|$$

$$= \frac{1}{4}|x+2|$$

For absolute convergence we need

$$\frac{1}{4}|x+2| < 1$$

$$|x+2| < 4 = R$$

∴ the radius of convergence is 4.

Lets determine the convergence of the endpoint

$$|x+2|<4$$

$$-4 < x+2 < 4$$

$$-6 < x < 2$$

Let $x=2$ then

$$\sum_{n=0}^{\infty} \frac{\cancel{(2+2)^n}}{\cancel{4^n}\sqrt[5]{n+3}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt[5]{n+3}}$$

$$= \frac{1}{\sqrt[3]{3}} + \sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n+3}}$$

$$a_n = \frac{1}{\sqrt[5]{n+3}} \geq \frac{1}{\sqrt[5]{n+3n}} = \frac{1}{\sqrt[5]{4}\, n^{1/5}} = b_n$$

$\sum_{n=1}^{\infty} b_n$ is a p-series where $p < \frac{1}{5}$, so it diverges

∴ by the comparison test $\sum_{n=1}^{\infty} a_n$ diverges. ∴ the power series does not converge at $x=2$

Let $x=-6$ then

$$\sum_{n=0}^{\infty} \frac{(-6+2)^n}{4^n\sqrt[5]{n+2}} = \sum_{n=0}^{\infty} \frac{(-4)^n}{4^n\sqrt[5]{n+2}} = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{4^n}}{\cancel{4^n}\sqrt[5]{n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt[5]{n+2}}$$

The above is an alternating series which can be written as

$$\sum_{n=0}^{\infty} (-1)^n b_n$$

where $b_n = \frac{1}{\sqrt[5]{n+2}}$

1) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[5]{n+2}} = 0$

2) $b_{n+1} \overset{?}{<} b_n$

$$\frac{1}{\sqrt[5]{n+3}} \overset{?}{<} \frac{1}{\sqrt[5]{n+2}}$$

$$\sqrt[5]{n+2} \overset{?}{<} \sqrt[5]{n+3}$$

$$n+2 < n+3$$

∴ by the alternating series test, the series converges.

∴ the interval of convergence is $[-6, 2)$

**Question 6.** *(5 marks)* Find the Taylor series for $f(x) = \sqrt{x}$ centered at 9.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f^{(0)}(9)}{0!}(x-9)^0 + \frac{f^{(1)}(9)(x-9)}{1!} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}1\cdot3\cdot5\cdots(2n-3)}{2^n\,3^{2n-1}}(x-9)^n$$

$$= 3 + \frac{1}{6}(x-9) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}1\cdot3\cdot5\cdots(2n-3)(x-9)^n}{2^n\,3^{2n-1}}$$

$f(x) = \sqrt{x}$ $\quad f(9) = \sqrt{9} = 3$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad f'(9) = \frac{1}{2}9^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{3}$

$f''(x) = \frac{1}{2}\frac{-1}{2}x^{-3/2}$ $\quad f''(9) = \frac{1}{2}\frac{-1}{2}\frac{1}{3^3}$

$f'''(x) = \frac{1}{2}\frac{-1}{2}\frac{-3}{2}x^{-5/2}$ $\quad f'''(9) = \frac{1}{2}\frac{-1}{2}\frac{-3}{2}\frac{1}{3^5}$

$f^{(4)}(x) = \frac{1}{2}\frac{-1}{2}\frac{-3}{2}\frac{-5}{2}x^{-7/2}$ $\quad f^{(4)}(9) = \frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdot\frac{1}{3^7}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^{n+1}}{2^n}1\cdot3\cdot5\cdots(2n-3)\,x^{-\frac{(2n-1)}{2}}$ $\quad f^{(n)}(9) = \frac{(-1)^{n+1}\,1\cdot3\cdot5\cdots(2n-3)}{2^n\cdot3^{2n-1}}$

**Question 7.** *(5 marks)* Determine whether the following series converges or diverges. Justify your answers

$$\sum_{n=1}^{\infty} \frac{1+\arcsin(\frac{1}{n})}{1+\arctan(n)}$$

Let $a_n = \dfrac{1+\arcsin(\frac{1}{n})}{1+\arctan(n)}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1+\arcsin(\frac{1}{n})}{1+\arctan(n)}$$

$$= \lim_{n\to\infty} \frac{1+\cancel{\arcsin}^{\to 0}(\cancel{\tfrac{1}{n}}^{\to 0})}{1+\cancel{\arctan(n)}_{\to \frac{\pi}{2}}}$$

$$= \frac{1}{1+\frac{\pi}{2}} \neq 0$$

$\therefore$ by the $n^{th}$ term divergence test the series diverges.

**Question 8.** *(10 marks)* Determine whether the foloowing series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2022}^{\infty} (-1)^n \frac{\ln(\ln n)+1}{n \ln n}$$

Lets verify for absolute convergence.

$$\sum_{n=2022}^{\infty} \left| (-1)^n \frac{\ln(\ln n)+1}{n \ln n} \right| = \sum_{n=2022}^{\infty} \frac{\ln(\ln n)+1}{n \ln n}$$

Lets apply the integral test.

Let $f(x) = \dfrac{\ln(\ln x)+1}{x \ln x}$

- $f(x)$ is positive on $[2022, \infty)$
- $f(x)$ is continuous on $[2022, \infty)$
- $f(x)$ is decreasing on $[2022, \infty)$ since

$$f'(x) = \frac{\frac{1}{\ln x}\cdot\frac{1}{x}\cdot x\ln x - (\ln(\ln x)+1)(\ln x + x\frac{1}{x})}{(x\ln x)^2}$$

$$= \frac{1 - \ln x \ln(\ln x) - \ln(\ln x) - \ln x - 1}{(x \ln x)^2}$$

$$= \frac{-\ln x \ln(\ln x) - \ln(\ln x) - \ln x}{(x\ln x)^2} < 0$$

$$\int_{2022}^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_{2022}^{b} \frac{\ln(\ln x)+1}{x \ln x}\,dx$$

$u = \ln(\ln x)+1$

$du = \frac{1}{\ln x}\cdot\frac{1}{x}$

$u(b) = \ln(\ln b)+1$

$u(2022) = \ln(\ln 2022)+1 = C$

$$= \lim_{b\to\infty} \int_{C}^{u(b)} u\,du$$

$$= \lim_{b\to\infty} \left[\frac{u^2}{2}\right]_C^{u(b)}$$

$$= \lim_{b\to\infty} \left[\frac{(u(b))^2}{2} - \frac{C^2}{2}\right]$$

$$= \lim_{b\to\infty} \left[\frac{\overbrace{(\ln(\ln b)+1)^2}^{\to\infty}}{2} - \frac{C^2}{2}\right]$$

$\therefore \int_{2022}^{\infty} f(x)\,dx$ diverges

$\therefore$ by integral test the series diverges

$\therefore$ the series is not absolutely convergent

Lets verify for conditional convergence

Let $\sum_{n=2022}^{\infty} (-1)^n b_n$ where $b_n = \dfrac{\ln(\ln n)+1}{n \ln n}$

Using the alternating series test

- $b_{n+1} < b_n$

since $f(n+1) < f(n)$

because $f'(x) < 0$ on $[2022, \infty)$

- $\lim_{x\to\infty} f(x)$

$$= \lim_{x\to\infty} \frac{\ln(\ln x)+1}{x\ln x} \quad \text{I.f. } \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{\ln x + x\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{1}{x\ln x}\cdot\frac{1}{(\ln x+1)}$$

$= 0$

$\therefore \lim_{n\to\infty} b_n = 0$

$\therefore$ by the alternating test the series converges

$\therefore$ the series is conditionally convergent.

**Bonus Question a.** *(3 marks)* Using the $K(\epsilon)$ definition show that the sequence $(-1)^n$ is divergent

**Bonus Question b.** *(3 marks)* Using the $K(\epsilon)$ definition show that given the sequence $a_n$ and $L \in \mathbb{R}$ such that there exists a $k \in \mathbb{N}$ such that for all $n \geq k$, $a_n = L$ then $a_n$ converges to $L$.

**Bonus Question c.** *(5 marks)* Using the $K(\epsilon)$ definition show that given the convergent sequence $a_n$ and $c \in \mathbb{R}$ then

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$