

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

**Question 1.**<sup>1</sup> (1 mark each) Complete each of the following sentences with MUST, MIGHT, or CANNOT.

- If  $\vec{v}$  is in  $\text{Span}(\{\vec{u}\})$ , then  $\vec{u}$  might be in  $\text{Span}(\{\vec{v}\})$ .
- If  $A\mathbf{x} = \mathbf{b}$  has two distinct solutions then the columns of  $A$  must be linearly dependent
- The columns of an elementary matrix must form a linearly independent set.
- If the set  $\{\vec{u}, \vec{v}, \vec{w}\}$  spans a plane in  $\mathbb{R}^3$ , then the set  $\{\vec{u}, \vec{v}\}$  might span the same plane.
- If  $\{\vec{u}, \vec{v}\}$  is a basis for a subspace  $S$ , then the set  $\{6\vec{u} + 3\vec{v}, 10\vec{u} + 5\vec{v}\}$  must also be a basis for  $S$ .

**Question 2.**<sup>1</sup> (1 mark each)

- Suppose that  $(3, -2, 7)$  and  $(-2, a, b)$  is linearly independent then a possibility for  $(a, b)$  is  $(a, b) = \underline{(0, 0)}$ .
- The vector space of all skew-symmetric  $n \times n$  matrices has dimension  $\frac{\sum_{i=1}^n i = n(n-1)}{2}$ .

**Question 3.**<sup>1</sup> Consider the set  $H = \{A \mid A \in M_{2 \times 2} \text{ and } [1 \ 3] A [1 \ 3]^T = 0\}$ .

- (1 marks) Find two vectors of  $H$ .
- (4 marks) Show that  $H$  is a subspace of  $M_{2 \times 2}$ .

b) lets apply the subspace test

① Closure under addition.

Let  $A, B \in H$  then  $A+B \in H$

$$\begin{aligned} \text{since } [1 \ 3](A+B)[1 \ 3]^T &= ([1 \ 3]A + [1 \ 3]B)[1 \ 3]^T \\ &= [1 \ 3]A[1 \ 3]^T + [1 \ 3]B[1 \ 3]^T \\ &= 0 + 0 \quad \text{since } A, B \in H \\ &= 0 \end{aligned}$$

② Closure under scalar multiplication

Let  $A \in H$  and  $k \in \mathbb{R}$  then  $kA \in H$

$$\begin{aligned} \text{since } [1 \ 3](kA)[1 \ 3]^T &= k [1 \ 3]A[1 \ 3]^T \\ &= k(0) \quad \text{since } A \in H \\ &= 0 \end{aligned}$$

∴  $H$  is a subspace of  $M_{2 \times 2}$  since it is closed under addition and scalar multiplication.

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$[1 \ 3] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0$$

$$[1 \ 3] \begin{bmatrix} a+3b \\ c+3d \end{bmatrix} = 0$$

$$a+3b+3(c+3d) = 0$$

$$a+3b+3c+9d = 0$$

$$-3b-3c-9d = a$$

$$\text{then } A = \begin{bmatrix} -3b-3c-9d & b \\ c & d \end{bmatrix} \in H$$

$$\text{a) let } b=c=d=0 \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$$

$$\text{let } b=c=d=1 \quad A = \begin{bmatrix} -15 & 1 \\ 1 & 1 \end{bmatrix} \in H$$

**Question 4.<sup>2</sup>** Let  $V = \{(a, b) \mid a, b \in \mathbb{R}, b > 0\}$ . And the addition in  $V$  is defined by  $(a, b) \oplus (c, d) = (ad + bc, bd)$  and scalar multiplication in  $V$  is defined by  $t \odot (a, b) = (tab^{t-1}, b^t)$

a. (1 mark)  $(4, 2) \oplus (-5, 1) = (4(1) + 2(-5), 2(1)) = (-6, 2)$

b. (1 mark)  $-2 \odot (1, 2) = (-2(1)(2)^{-2-1}, 2^{-2}) = (-\frac{2}{8}, \frac{1}{4})$

c. (3 marks) Demonstrate whether the 4th axiom of vector spaces holds. That is, does the zero vector exist.

<p>Let <math>\underline{z} = (z_1, z_2)</math> and <math>\underline{u} = (a, b) \in V</math></p> <p><math>\underline{u} + \underline{z} = \underline{u}</math></p> <p><math>(a, b) + (z_1, z_2) = (a, b)</math></p> <p><math>(az_2 + bz_1, bz_2) = (a, b)</math></p> <p>① <math>az_2 + bz_1 = a</math></p> <p>② <math>bz_2 = b</math></p> <p>since <math>b &gt; 0</math> <math>z_2 = 1</math></p>	<p>sub into ① <math>a + bz_1 = a</math></p> <p><math>bz_1 = 0</math></p> <p><math>z_1 = 0</math> since <math>b &gt; 0</math></p> <p><math>\therefore \underline{z} = (0, 1)</math> and <math>\underline{z} \in V</math> since <math>1 &gt; 0</math>.</p>
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**Question 5.<sup>3</sup>** (7 marks) Find a basis for all vectors of the form  $(a, b, c, d)$  where  $c = a + b$  and  $d = a - b$  and state its dimension. Also express  $(2, 3, -1, 5)$  relative to the basis found, if possible.

Let  $W = \{(a, b, c, d) \mid d = a + b \text{ and } c = a - d\}$

Let  $\underline{x} = (a, b, a + b, a - d) \in W$

$= (a, 0, a, a) + (0, b, b, -b)$

$= a \underbrace{(1, 0, 1, 1)}_{\underline{b}_1} + b \underbrace{(0, 1, 1, -1)}_{\underline{b}_2}$

$\therefore \mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$  span  $W$

And  $\mathcal{B}$  is linearly independent since  $\underline{b}_1$  and  $\underline{b}_2$  are not multiples of each other.

$\therefore \mathcal{B}$  is a basis of  $W$ .

$\therefore \dim(W) = 2$

$((2, 3, -1, 5))_{\mathcal{B}} = \text{undefined}$

since  $(2, 3, -1, 5) \notin W$

because  $d \neq a - d$   
 $5 \neq 2 - 3$

<sup>2</sup>From <http://www.math.uwaterloo.ca/~jmckinn/Math225/Week1/Lecture1e.pdf>

<sup>3</sup>From the assigned homework.

**Question 6.**<sup>3</sup> (4 marks) Prove: Let  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  be a basis for a vector space  $V$ . Show that  $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$  is also a basis, where  $\vec{u}_1 = \vec{v}_1$ ,  $\vec{u}_2 = \vec{v}_1 + \vec{v}_2$ , and  $\vec{u}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ .

Let's show that  $\text{span}(S) = \text{span}(S')$

①  $\text{span}(S) \subseteq \text{span}(S')$

$$\begin{aligned} \vec{v}_1 &\in \text{span}(S') \text{ since } \vec{v}_1 = 1\vec{u}_1 + 0\vec{u}_2 + 0\vec{u}_3 \\ \vec{v}_2 &\in \text{span}(S') \text{ " } \vec{v}_2 = -1\vec{u}_1 + 1\vec{u}_2 + 0\vec{u}_3 \\ \vec{v}_3 &\in \text{span}(S') \text{ " } \vec{v}_3 = 0\vec{u}_1 + (-1)\vec{u}_2 + 1\vec{u}_3 \end{aligned}$$

∴  $\text{span}(S) \subseteq \text{span}(S')$

②  $\text{span}(S') \subseteq \text{span}(S)$

$$\begin{aligned} \vec{u}_1 &\in \text{span}(S) \text{ since } \vec{u}_1 = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \\ \vec{u}_2 &\in \text{span}(S) \text{ " } \vec{u}_2 = 1\vec{v}_1 + 1\vec{v}_2 + 0\vec{v}_3 \\ \vec{u}_3 &\in \text{span}(S) \text{ " } \vec{u}_3 = 1\vec{v}_1 + 1\vec{v}_2 + 1\vec{v}_3 \end{aligned}$$

∴  $\text{span}(S') \subseteq \text{span}(S)$

∴  $\text{span}(S') = \text{span}(S)$

Since the number of vectors in  $S'$  =  $\dim(\text{span}(S))$  and  $S'$  spans the  $\text{span}(S)$  it follows that  $S'$  is linearly independent.

∴  $S'$  is a basis of  $\text{span}(S) = V$ .

**Question 7.** (4 marks) Determine whether  $\left\{ \underbrace{\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 2 & -1 \\ 7 & 3 \end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 3 & 1 \\ -3 & 5 \end{bmatrix}}_{M_3} \right\}$  is linearly independent.

$0 = c_1 M_1 + c_2 M_2 + c_3 M_3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ -3 & 7 & 3 \\ 5 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & -1 & -1 & 0 \\ -3 & 7 & 3 & 0 \\ 5 & 3 & 5 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} -2R_1 + R_2 \rightarrow R_2 \\ 3R_1 + R_3 \rightarrow R_3 \\ -5R_1 + R_4 \rightarrow R_4 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 13 & 12 & 0 \\ 0 & -7 & -10 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} 5R_3 \rightarrow R_3 \\ 5R_4 \rightarrow R_4 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 65 & 60 & 0 \\ 0 & -35 & -50 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} 13R_2 + R_3 \rightarrow R_3 \\ -7R_2 + R_4 \rightarrow R_4 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} 3R_4 + R_1 \rightarrow R_1 \\ -7R_4 + R_2 \rightarrow R_2 \\ -31R_4 + R_3 \rightarrow R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} -1/5 R_2 \rightarrow R_2 \\ R_3 \leftrightarrow R_4 \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{matrix} -R_2 + R_1 \rightarrow R_1 \\ -R_3 \rightarrow R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ only trivial sol.

∴ set is linearly independent.

**Question 8.** (4 marks) In any vector space  $V$ , for any  $\underline{u}, \underline{v}, \underline{w} \in V$  prove that if  $\underline{v} + \underline{w} = \underline{u} + \underline{w}$  then  $\underline{v} = \underline{u}$ . Show every step, justify every step, and cite the axiom(s) used!!!

$$\underline{v} + \underline{w} = \underline{u} + \underline{w}$$

$$\underline{v} + \underline{w} + \underline{w}' = \underline{u} + \underline{w} + \underline{w}'$$

$$\underline{v} + (\underline{w} + \underline{w}') = \underline{u} + (\underline{w} + \underline{w}')$$

$$\underline{v} + \underline{0} = \underline{u} + \underline{0}$$

$$\underline{v} = \underline{u}$$

the additive inverse of  $\underline{w}$  exists  
by axiom 5 (add  $\underline{w}'$  on both sides)  
associativity, axiom 3

axiom 5

axiom 4.

**Bonus Question.** (1+5 marks) State and prove the Exchange Lemma.