Dawson College: Winter 2022: Linear Algebra (Science): 201-NYC-05-S8: **Test 1, part 1 of 2** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.[1] *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. A system of linear equations with more equations than variables might have a unique solution.

b. If the 3×3 coefficient matrix A has a RREF with a single leading one, then the system $A\mathbf{x} = \mathbf{b}$ might be inconsistent when $\mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

c. If the bottom row of a matrix in reduced row-echelon form contains all 0's to the left of the vertical bar and a nonzero entry to the right, then the system must have no solution.

d. If A is a square matrix and $A^4 - 2A^2 + A = 0$, then A might be invertible.

e. Given an $n \times n$ matrix A. If the system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^n$, then the system $A\mathbf{x} = \mathbf{0}$ cannot have non-trivial solutions.

f. The expression $(B - A)(B + A)$ must be equal to $B^2 - A^2$ if A and B are commutative.

g. If E_1, E_2 are elementary matrices, then E_1E_2 might also be an elementary matrix.

h. If matrix AB is invertible, then B might be invertible.

Question 2.[1] *(5 marks)* Given X, A, B, and C are $n \times n$ matrices, solve the following equation for X. Assume any necessary matricesto be invertible.

$$2X^T + A = (XB - C)^T$$

$$2X^T + A = (XB)^T - C^T$$

$$2X^T + A = B^T X^T - C^T$$

$$C^T + A = -2X^T + B^T X^T$$

$$C^T + A = (-2I + B^T)X^T$$

$$(-2I + B^T)^{-1}(C^T + A) = (-2I + B^T)^{-1}(-2I + B^T)X^T$$

$$(-2I + B^T)^{-1}(C^T + A) = X^T$$

$$\left((-2I + B^T)^{-1}(C^T + A)\right)^T = (X^T)^T$$

$$(C^T + A)^T\left((-2I + B^T)^{-1}\right)^T = X$$

$$\left((C^T)^T + A^T\right)\left((-2I + B^T)^T\right)^{-1} = X$$

$$(C + A)(-2I + B)^{-1} = X$$

Question 3.[2] *(5 marks)* Find **all** matrices A, if any, such that $\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{B} A = I_2$

For the product of BA to give a 2×2 matrix, A must be a 3×2 matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ then $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a+c+e & b+d+f \\ c+e & d+f \end{bmatrix}$

$1 = a + c + e$

$0 = c + e$

$0 = b + d + f$

$1 = d + f$

$$\begin{array}{ccccccc} a & b & c & d & e & f & \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \sim_{R_2 \leftrightarrow R_3} \left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} -R_3 + R_1 \to R_1 \\ -R_4 + R_2 \to R_2 \end{array} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Let $e = t$, $f = s$ where $t, s \in \mathbb{R}$

$\therefore A = \begin{bmatrix} 1 & -1 \\ -t & 1-s \\ t & s \end{bmatrix}$ $t, s \in \mathbb{R}$

[1] From or modified from a John Abbott final examination or WeBWorK

[2] From a Dawson final examination

Question 4. Given

$$A = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 2 & -2 & 0 & 3 \\ 3 & 0 & 0 & -2 \\ 5 & -2 & -1 & 0 \end{bmatrix} \underset{\substack{R_1 \leftrightarrow R_2 \\ 2R_1 \to R_3 \\ 2R_1 \to R_4}}{\sim} \begin{bmatrix} 2 & -2 & 0 & 3 \\ 0 & 0 & -1 & -1 \\ 6 & 0 & 0 & -4 \\ 10 & -4 & -2 & 0 \end{bmatrix} \underset{\substack{-3R_1+R_3 \to R_3 \\ -5R_1+R_4 \to R_4}}{\sim} \begin{bmatrix} 2 & -2 & 0 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 6 & 0 & -13 \\ 0 & 6 & -2 & -15 \end{bmatrix}$$

a. *(4 marks)* Find two row echelon form of the matrix A.

b. *(2 marks)* Suppose that A is the augmented matrix of a linear system. Write the corresponding system of linear equations. Use one of the REF from part a. to find the solution of the system.

c. *(1 mark)* Express A as a product of elementary matrices, if possible. Justify.

d. *(1 mark)* Dethermine whether A is symmetric, skew-symmetric or neither. Justify.

$$\underset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 2 & -2 & 0 & 3 \\ 0 & 6 & 0 & -13 \\ 0 & 0 & -1 & -1 \\ 0 & 6 & -2 & -15 \end{bmatrix} \underset{\substack{-R_3 \to R_3 \\ -R_2 + R_4 \to R_4}}{\sim} \begin{bmatrix} 2 & -2 & 0 & 3 \\ 0 & 6 & 0 & -13 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \underset{2R_3 + R_4 \to R_4}{\sim} \begin{bmatrix} 2 & -2 & 0 & 3 \\ 0 & 6 & 0 & -13 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\substack{\frac{1}{2}R_1 \to R_1 \\ \frac{1}{6}R_2 \to R_2}}{\sim} \begin{bmatrix} 1 & -1 & 0 & 3/2 \\ 0 & 1 & 0 & -13/6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{a REF of } A$$

$$\underset{R_3 + R_2 \to R_2}{\sim} \begin{bmatrix} 1 & -1 & 0 & 3/2 \\ 0 & 1 & 1 & -7/6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{an other REF of } A$$

b) From the augmented matrix we have

① $x_1 - x_2 = 3/2$

② $x_2 = -13/6$

③ $x_3 = 1$

Using back substitution,

$x_2 = -\frac{13}{6}$

sub ④ into ①

$x_1 - \left(-\frac{13}{2}\right) = \frac{3}{2}$

$x_1 = \frac{3}{2} - \frac{13}{6} = \frac{-4}{6} = \frac{-2}{3}$

$\therefore (x_1, x_2, x_3) = \left(-\frac{2}{3}, \frac{-13}{2}, 1\right)$

c) Not expressible as a product of elem. mat. by the equiv. thm. since the RREF is not I.

d)

$$A^T = \begin{bmatrix} 0 & 2 & 3 & 5 \\ 0 & -2 & 0 & -2 \\ -1 & 0 & 0 & -1 \\ -1 & 3 & -2 & 0 \end{bmatrix} \begin{matrix} \neq A \therefore \text{not symmetric} \\ \neq -A \therefore \text{not skew-symmetric} \end{matrix}$$

Question 5a.[1] *(5 marks)* Let $A = \begin{bmatrix} 1 & 0 & 5 \\ -1 & 3 & 1 \\ 1 & k & 12 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 8 \\ h \end{bmatrix}$. Find the value(s) of h and k, if possible, for which the equation $A\mathbf{x} = \mathbf{b}$ has:

a. a unique solution,
b. infinitely many solutions,
c. no solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ -1 & 3 & 1 & 8 \\ 1 & k & 12 & h \end{array}\right] \sim \begin{array}{l} R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 0 & 3 & 6 & 6 \\ 0 & k & 7 & h+2 \end{array}\right] \sim \tfrac{1}{3}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & k & 7 & h+2 \end{array}\right]$$

$$\sim -kR_2+R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 7-2k & h+2-2k \end{array}\right]$$

a) # var = # leading entries
In order to have 3 leading entries $7-2k \neq 0$
$7 \neq 2k$
$\frac{7}{2} \neq k$

b) #var > #leading entries (no leading entry in constant col.)
So $7-2k=0$ and $h+2-2k=0$
$k=\frac{7}{2}$
which implies $h = 2k-2 = 2\left(\frac{7}{2}\right)-2 = 5$

c) Leading entry in the constant col.
So $7-2k=0$ and $h+2-2k \neq 0$. Based on the above
this occurs when $k=\frac{7}{2}$ and $h \neq 5$.

Question 5b. *(3 marks)* Find the solution set in the case where the above system has infinitely many solutions. Give a particular solution. And the particular solution when the second variable is equal to zero.

To have infinitely many $k=\frac{7}{2}$ and $h=5$, $\left[\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Let $x_3 = t$, $t \in \mathbb{R}$

$x_1 = -2-5t$
$x_2 = 2-2t$

$\therefore (x_1,x_2,x_3) = (-2-5t, 2-2t, t)$, $t \in \mathbb{R}$
and a particular solution when $t=0$, $(x_1,x_2,x_3) = (-2,2,0)$

and when $0 = x_2$
$0 = 2-2t$
$t = 1$
$\therefore (x_1,x_2,x_3) = (-7,0,1)$.

Question 6. *(3 marks)* A square matrix A is *idempotent* if $A^2 = A$. Prove that if A is idempotent then $(I-2A) = (I-2A)^{-1}$.

Premise:
- A is idempotent, that is, $A^2 = A$

Conclusion:
- $(I-2A)$ is invertible and its own inverse.

Need to show that there exist a B s.t. $(I-2A)B = I$
$B(I-2A) = I$

Let $B = (I-2A)$

$$\begin{aligned} (I-2A)B &= (I-2A)(I-2A) \\ &= I \cdot I - I(2A) - (2A)\cdot I + (2A)(2A) \\ &= I - 4A + 4A^2 \\ &= I - 4A + 4A = I \end{aligned}$$

$\therefore (I-2A)^{-1} = (I-2A)$

Question 7. *(5 marks)* Express $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and its inverse as a product of elementary matrices, if possible. Justify.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim_{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \sim^{R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \sim_{-\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since the RREF of A is I, A is invertible. It also follows that

$$E_3 E_2 E_1 A = I$$

where

$$I_2 \sim_{-3R_1+R_2} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = E_1 \Rightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$I_2 \sim_{R_2+R_1 \to R_1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = E_2 \Rightarrow E_2^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$I_2 \sim_{-\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} = E_3 \Rightarrow E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

and $E_3 E_2 E_1 A A^{-1} = I A^{-1}$ by a
$E_3 E_2 E_1 = A^{-1}$

and $(E_3 E_2 E_1)^{-1} = (A^{-1})^{-1}$
$E_1^{-1} E_2^{-1} E_3^{-1} = A$

Question 8.[3] *(5 marks)* Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n \times n$ matrix. Prove that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $(QA)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

($\Rightarrow$) premise:
$Ax = 0$ has only the trivial sol.
Q is invertible

conclusion:
$(QA)x = 0$ has only the trivial sol.

Since $Ax = 0$ has only the trivial sol. A is invertible by the equivalence thm.

QA is then invertible since its the product of two invertible matrix.

Again by the equiv. thm. $(QA)x = 0$ has only the trivial sol. since QA is invertible.

[$\Leftarrow$] premise:
- $(QA)x = 0$ has only the triv. sol.
- Q is invertible

conclusion:
$Ax = 0$ has only the trivial sol.

It follows that QA is invertible by the equiv. thm since a. So

$$\underbrace{(QA)^{-1}Q}A = I$$

is the inverse of A by a thm seen in class. Hence A is invertible and by the equiv thm $Ax = 0$ has only the trivial sol.

Bonus Question. *(5 marks)* Prove: $(aA)(bB) = abAB$ where A and B are matrices such that their multiplication is defined and $a, b \in \mathbb{R}$

[3]From assigned homework.