Dawson College: Winter 2022: Linear Algebra (Science): 201-NYC-05-S8: **Test 2, part 1 of 2** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.[1] *(1 mark per blank)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. If A is a product of elementary matrices, then $\det(A)$ cannot equal zero..

b. Let A be a 3×3 matrix, and let B be a 4×4 matrix. If the leading ones of the RREF of A is equal to those of the RREF of B, then $\det(B)$ must equal zero and $det(A)$ might equal zero.

c. Two lines in $\mathbb{R}^3$ that are both perpendicular to a third line might be parallel.

d. If $\vec{u}$ and $\vec{v}$ are nonzero vectors in $\mathbb{R}^3$, then $(\vec{u} \times \vec{v}) \cdot \vec{u}$ must be equal to 0.

e. Let $\vec{u}$ be parallel to $\vec{x}$, and let $\vec{v}$ be parallel to $\vec{y}$. Then $\vec{u} + \vec{v}$ might be parallel to $\vec{x} + \vec{y}$.

f. The vector $\vec{u} \times (\vec{v} \times \vec{w})$ might be a solution of $\vec{v} \cdot \vec{x} = 0$ and $\vec{w} \cdot \vec{x} = 0$.

Question 2. *(1 mark per blank)* Given A an $n \times n$ matrix and k a non-zero scalar.

a. If A is an elementary matrix obtained by interchanging two rows then $\det(A) =$ -1.

b. If A is a matrix which is obtained by multiplying each row of the identity by the number of the row then $\det(A) =$ $1 \cdot 2 \cdot 3 \cdots n = n!$.

c. If A is an elementary matrix obtained by multiplying one row by k then $\det(A) =$ k.

Question 3.[2] *(5 marks)* If A and B are invertible matrices of the same size show that

$$\text{adj}\,(AB) = \text{adj}\,(B)\,\text{adj}\,(A)$$

· Since A and B are invertible we have that $A^{-1} = \frac{1}{\det A}\,\text{adj}\,A$

$B^{-1} = \frac{1}{\det B}\,\text{adj}\,B$

It also follows that since A and B are invertible that AB is invertible.

$$(AB)^{-1} = \frac{1}{\det AB}\,\text{adj}\,B$$

$$\text{adj}\,AB = (\det AB)(AB)^{-1}$$

$$\text{adj}\,AB = \det A \det B\; B^{-1}A^{-1}$$

$$= (\det B)\,B^{-1}(\det A)\,A^{-1}$$

$$= \text{adj}\,B\;\text{adj}\,A$$

Question 4. *(5 marks)* Find the determinant of the matrix A.

$$A = \begin{bmatrix} 2 + 2\text{trace}(A) & \det(A) & 1 \\ -2 & 3 & 1 \\ 2 & 5 & 0 \end{bmatrix}$$

$\text{trace}(A) = 2 + 2\,\text{trace}(A) + 3 + 0$

$5 = -\text{trace}(A)$

$-5 = \text{trace}(A)$

$$A = \begin{bmatrix} -8 & \det(A) & 1 \\ -2 & 3 & 1 \\ 2 & 5 & 0 \end{bmatrix}$$

$$\det(A) = a_{13}C_{13} + a_{23}C_{23} + \underbrace{a_{33}C_{33}}^{0}$$

$$\det(A) = 1\begin{vmatrix} -2 & 3 \\ 2 & 5 \end{vmatrix} - (1)\begin{vmatrix} -8 & \det(A) \\ 2 & 5 \end{vmatrix}$$

$$\det(A) = -16 + 40 + 2\det(A)$$

$24 = -\det A$

$-24 = \det A$

[1] From or modified from a John Abbott final examination

[2] From a Dawson College final examination

Question 5.[1] *(5 marks)* Given that $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = 10$ and $A = \begin{bmatrix} 3g+a & 3h+b & 2 & 3i+c \\ d+2a & e+2b & 3 & f+2c \\ a & b & 4 & c \\ 0 & 0 & 5 & 0 \end{bmatrix}$

a. *(5 marks)* Find $\det(A)$.

b. *(3 marks)* Using Cramer's Rule find x_1 and x_3 for $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 2a & 3a & 4a & 5a \end{bmatrix}^T$

a) $|A| = \underbrace{a_{41}C_{41}}_{0} + \underbrace{a_{42}C_{42}}_{0} + a_{43}C_{43} + \underbrace{aC_{44}}_{0}$

$= 5(-1)^{4+3} \begin{vmatrix} 3g+a & 3h+b & 3i+c \\ d+2a & e+2b & f+2c \\ a & b & c \end{vmatrix}$

$= -5 \begin{vmatrix} 3g & 3h & 3i \\ d & e & f \\ a & b & c \end{vmatrix}$ after $-R_3+R_1 \to R_1$, $-2R_3+R_2 \to R_2$

$= -5(3) \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$ after $\frac{1}{3}R_1 \to R_1$

$= -5(3)(-1) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 15(10) = 150$

b) $|A_1| = \begin{vmatrix} 2a & 3h+b & 2 & 3i+c \\ 3a & e+2b & 3 & f+2c \\ 4a & b & 4 & c \\ 5a & 0 & 5 & 0 \end{vmatrix} = 0$ since $C_1 = aC_3$

$\therefore x_1 = \frac{|A_1|}{|A|} = \frac{0}{150} = 0$

$|A_3| = \begin{vmatrix} 3g+a & 3h+b & 2a & 3i+c \\ d+2a & e+2b & 3a & f+2c \\ a & b & 4a & c \\ 0 & 0 & 5a & 0 \end{vmatrix}$

$= a|A|$ after $\frac{1}{a}R_3 \to R_3$, note if $a=0$ then $|A_3|=0$ and $x_3 = 0$

$= a150$

$\therefore x_3 = \frac{|A_3|}{|A|} = \frac{a150}{150} = a$

Question 6.[1] *(3 mark)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

If A is a skew-symmetrix $n \times n$ matrix where n is odd then $\det(A) = 0$.

True,

Since A is skew-symmetric

$A^T = -A$

$\det(A^T) = \det(-A)$

$\det(A) = (-1)^n \det(A)$

$\det(A) = -\det(A)$ since n is odd

$\therefore \det(A) = 0$

Question 7.[1] *(3 marks)* Show that if $\text{proj}_{\vec{v}}\,\vec{u} = \text{proj}_{\vec{v}}\,\vec{w}$, then $\vec{u} - \vec{w}$ is orthogonal to $\vec{v}$.

From the premise: $\frac{\underline{u}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\underline{v} = \frac{\underline{w}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\underline{v}$

$(\underline{u}\cdot\underline{v})\underline{v} = (\underline{w}\cdot\underline{v})\underline{v}$

$((\underline{u}\cdot\underline{v}) - (\underline{w}\cdot\underline{v}))\,\underline{v} = \underline{0}$

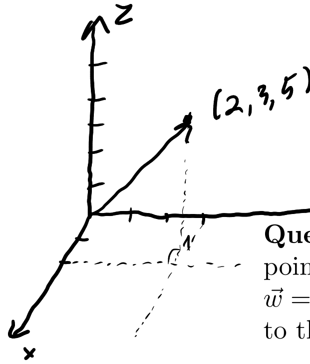
If $\underline{v} = \underline{0}$ then $\underline{u} - \underline{v}$ is orthogonal to $\underline{v}$.

If $\underline{v} \neq \underline{0}$ then $\underline{u}\cdot\underline{v} - \underline{w}\cdot\underline{v} = 0$

$(\underline{u} - \underline{w})\cdot\underline{v} = 0$

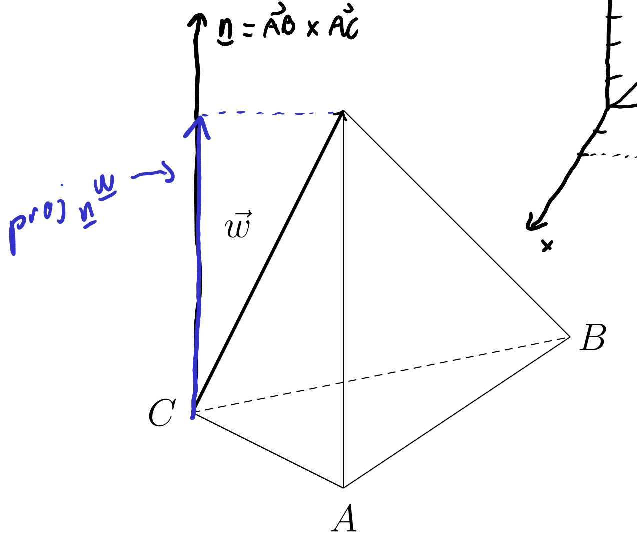
$\therefore \underline{u} - \underline{w}$ is orthogonal to $\underline{v}$

Question 8. *(2 marks)* Sketch $\vec{v} = (2, 3, 5)$ as shown in class, include and label the axes.



Question 9. *(5 marks)* Given the terahedron determined by the points $A(2, -1, -1)$, $B(2, -1, -2)$, $C(0, 8\sqrt{5}-1, 0)$ and the vector $\vec{w} = (4, 1, 3)$. Note that from the diagram, $\vec{w}$ is not perpendicular to the base.

Find the volume of the tetrahedron. *(Hint: the volume of a tetrahedron is equal to one third of the area of the base times the height.)*



$\vec{AB} = (2,-1,-2) - (2,-1,-1) = (0,0,-1)$

$\vec{AC} = (0, 8\sqrt{5}-1, 0) - (2,-1,-1) = (-2, 8\sqrt{5}, 1)$

$\underline{n} = \vec{AB} \times \vec{AC} = \left(\begin{vmatrix} 0 & 8\sqrt{5} \\ -1 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & -2 \\ 0 & 8\sqrt{5} \end{vmatrix} \right)$

$\begin{matrix} 0 & -2 \\ 0 & 8\sqrt{5} \\ -1 & 1 \end{matrix} \quad = (8\sqrt{5}, +2, 0)$

$V = \frac{1}{3}$ base $\cdot$ height

$= \frac{1}{3} 9 \|proj_{\underline{n}} w\|$

$= 3 \left\| \frac{\underline{n} \cdot w}{\underline{n} \cdot \underline{n}} \underline{n} \right\|$ where $\underline{n} = \vec{AB} \times \vec{AC}$ and $C(0, 8\sqrt{5}-1, 0)$

$= 3 \left\| \frac{\underline{n} \cdot w}{\|n\|^2} \underline{n} \right\|$

$= 3 \, |\underline{n} \cdot \underline{w}| \frac{\|n\|}{\|n\|^2}$

$= \frac{3}{\|n\|} |n \cdot \underline{w}|$

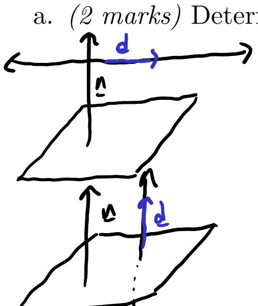
$= \frac{3}{\sqrt{(8\sqrt{5})^2 + 2^2 + 0^2}} |(8\sqrt{5}, 2, 0) \cdot (4,1,3)|$

$= \frac{3}{\sqrt{64(5)+4}} |32\sqrt{5} + 2|$

$= \frac{1}{6}(32\sqrt{5} + 2)$

Question 10. Given the plane $x + y + z = 0$ and the line $(x,\ y,\ z) = (1 + t, 2 + 2t, 3 + 3t)$ where $t \in \mathbb{R}$.

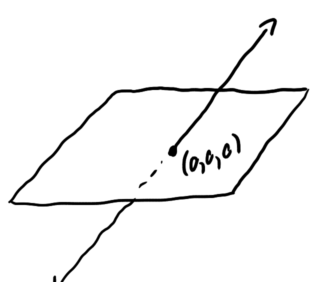
a. *(2 marks)* Determine whether the line is perpendicular to the plane, parallel or neither. Justify



For the plane and line to be parallel, $\underline{d}$ must be $\perp$ to $\underline{n}$.
$\underline{d} \cdot \underline{n} = (1,2,3) \cdot (1,1,1) = 6 \neq 0 \quad \therefore$ not parallel.

For the plane and line to be perpendicular, $\underline{d}$ must be $\parallel$ to $\underline{n}$.
The vectors are not multi. of each other. $\therefore$ not perpendicular

b. *(2 marks)* Find the point of intersection between the line and the plane if it exists.



$(1+t) + (2+2t) + (3+3t) = 0$

$6 + 6t = 0$

$t = -1$

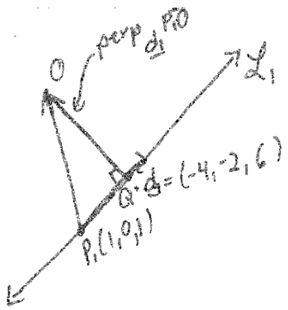
$\therefore$ intersection when $t = -1$

$\therefore (x,y,z) = (1-1, 2+2(-1), 3+3(-1))$

$= (0,0,0)$

Question 11.[1] Given $\mathcal{L}_1 : \vec{x} = (1,0,1) + t(-4,-2,6)$, where $t \in \mathbb{R}$.

a. *(4 marks)* Find an equation for the line through the origin that intersects $\mathcal{L}_1$ at a right angle.



$\vec{P_1O} = (0,0,0) - (1,0,1) = (-1,0,-1)$

$perp_{d_1}\vec{P_1O} = \vec{P_1O} - proj_{d_1}\vec{P_1O}$

$= (-1,0,-1) - \frac{\vec{P_1O}\cdot d_1}{d_1 \cdot d_1} d_1$

$= (-1,0,-1) - \frac{(-1,0,-1)\cdot(-4,-2,6)}{(-4,-2,6)\cdot(-4,-2,6)}(-4,-2,6)$

$= (-1,0,-1) - \frac{4-6}{16+4+36}(-4,-2,6)$

$= (-1,0,-1) - \frac{-2}{56}(-4,-2,6)$

$= (-1,0,-1) + \frac{1}{28}(-4,-2,6) = \left(\frac{-8}{7}, \frac{-1}{14}, \frac{-11}{14}\right)$

$\therefore$ $\underline{X} = (0,0,0) + t\left(\frac{-8}{7}, \frac{-1}{14}, \frac{-11}{14}\right)$

where $t \in \mathbb{R}$

b. *(2 marks)* Find the distance between the origin and $\mathcal{L}_1$.

$d = \|perp_{d_1}\vec{P_1O}\| = \left\|\left(\frac{-8}{7}, \frac{-1}{14}, \frac{-11}{14}\right)\right\| = \left\|\frac{1}{14}(-16,-1,-11)\right\| = \frac{1}{14}\sqrt{(-16)^2+(-1)^2+(-11)^2}$

$= \frac{1}{14}\sqrt{378}$

c. *(2 marks)* Find the closest point on $\mathcal{L}_1$ to the origin.

$\vec{QO} = perp_{d_1}\vec{P_1O}$

$-\underline{Q} = \left(\frac{-8}{11}, \frac{-1}{14}, \frac{-11}{14}\right)$

$\therefore$ $Q = \left(\frac{8}{11}, \frac{1}{14}, \frac{11}{14}\right)$

c. *(2 marks)* Find the plane that contains $\mathcal{L}_1$ and the origin.

$\underline{X} = (0,0,0) + s\underline{d_1} + t\, perp_{d_1}\vec{P_1O}$

$= s(-4,-2,6) + t\left(\frac{-8}{7}, \frac{-1}{14}, \frac{-11}{14}\right)$ $t \in \mathbb{R}$

Question 12.[3]

a. *(1 marks)* Find a homogeneous linear system of two equations in three unknowns whose solution space consists of those vectors in $\mathbb{R}^3$ that are orthogonal to $\vec{a} = (-3,2,-1)$ and $\vec{b} = (0,-2,-2)$.

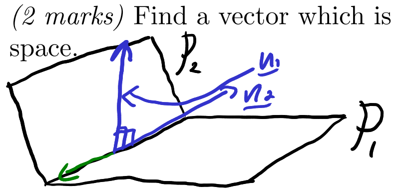
Let $\underline{x} = (x,y,z)$,

$\left.\begin{array}{l}\underline{x}\cdot\underline{a} = 0\\ \underline{x}\cdot\underline{b} = 0\end{array}\right\} \Rightarrow \begin{array}{l}-3x+2y-z=0\\ -2y-2z=0\end{array}$

b. *(2 marks)* What kind of geometric object is the solution space? Justify.

The solution space is a line since the system is comprised of two planes that are not parallel because their normals are not multiples of each other.

c. *(2 marks)* Find a vector which is parallel to the solution space without solving the system. Using that vector find the solution space.



$\underline{d} = n_1 \times n_2 = \left(\begin{vmatrix}2 & -2\\ -1 & -2\end{vmatrix}, -\begin{vmatrix}-3 & 0\\ -1 & -2\end{vmatrix}, \begin{vmatrix}-3 & 0\\ 2 & -2\end{vmatrix}\right) = (-6,-6,6)$

$\begin{array}{cc}-3 & 0\\ 2 & -2\\ -1 & -2\end{array}$

$\therefore$ $\underline{X} = (0,0,0) + t\underline{d}$ $t \in \mathbb{R}$

Bonus Question. *(5 marks)* If $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $|\vec{u}\cdot\vec{v}| \le ||\vec{u}||||\vec{v}||$. *Hint: Analyse the squared norm of* $||\vec{u}||\vec{v} - ||\vec{v}||\vec{u}$ *and* $||\vec{u}||\vec{v} + ||\vec{v}||\vec{u}$.

[3]From the assigned homework.