Dawson College: Winter 2022: Linear Algebra (SCIENCE): 201-NYC-05-S8: **Test 3, part 1 of 2** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.[1] *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. Let $S = \{\vec{u}, \vec{v}\}$ be a set of vectors. If $\vec{w}$ is in Span(S), then $\vec{w}$ might be in S.

b. Let $\vec{u}, \vec{v}$, and $\vec{w}$ be distinct nonzero vectors in $\mathbb{R}^3$. If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, then $\vec{u} \cdot (\vec{v} \times \vec{w})$ cannot be equal to $\vec{u} \cdot (\vec{w} \times \vec{v})$.

c. If $\{\vec{a}, \vec{b}, \vec{c}\}$ is a linearly independent set in Span$(\{\vec{u}, \vec{v}, \vec{w}\})$, then $\{\vec{u}, \vec{v}, \vec{w}\}$ must be a linearly independent set.

d. If B has no column of zeros, but AB does, then the columns of A cannot be linearly independent.

e. If the column vectors of a square matrix A span all of $\mathbb{R}^3$, then the determinant of A cannot be zero.

Question 2.[1] *(1 mark each)*

a. Suppose that $(3, -2, 7)$ and $(-2, a, b)$ is linearly dependent then $(a, b) =$ $(4/3, -14/3)$.

b. The vector space of all symmetric $n \times n$ matrices has dimension $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Question 3.[1] Consider the subspace $H = \left\{ A \mid A \in \mathcal{M}_{2\times2} \text{ and } \begin{bmatrix} 1 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 3 \end{bmatrix}^T = \mathbf{0} \right\}$.

a. *(1 marks)* Find two vectors of H.

b. *(4 marks)* Find a basis for H.

c. *(1 mark)* State the dim(H).

d. *(2 marks)* Express $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$ relative to the basis found in part b., if possible.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} a+3b \\ c+3d \end{bmatrix} = 0$$

$$a + 3b + 3(c+3d) = 0$$

$$a + 3b + 3c + 9d = 0$$

$$-3b - 3c - 9d = a$$

then $A = \begin{bmatrix} -3b-3c-9d & b \\ c & d \end{bmatrix} \in H$

a) let $b = c = d = 0$ $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$

let $b = c = d = 1$ $A = \begin{bmatrix} -15 & 1 \\ 1 & 1 \end{bmatrix} \in H$

b) $A = \begin{bmatrix} -3b-3c-9d & b \\ c & d \end{bmatrix}$

$$= \begin{bmatrix} -3b & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -3c & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} -9d & 0 \\ 0 & d \end{bmatrix}$$

$$= b \underbrace{\begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}}_{M_1} + c \underbrace{\begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix}}_{M_2} + d \underbrace{\begin{bmatrix} -9 & 0 \\ 0 & 1 \end{bmatrix}}_{M_3}$$

Then $\beta = \{M_1, M_2, M_3\}$ spans H.

$$0 = c_1 M_1 + c_2 M_2 + c_3 M_3$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} -9 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ $\therefore$ β is linearly independent

$\therefore$ β is a basis

c) $\dim(H) = 3$

d) $\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} \notin H$ since if $b = 3, c = 2, d = 4$

then $a = -3(3) - 3(2) - 9(4) \neq 2$

$\therefore$ $\left(\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} \right)_\beta$ is undefined.

[1] From or modified from a John Abbott final examination

Question 4.[2] Let $V = \{(a, b) \mid a, b \in \mathbb{R}, b > 0\}$. And the addition in V is defined by $(a, b) \bigoplus (c, d) = (ad + bc, bd)$ and scalar multiplication in V is defined by $t \bigodot (a, b) = (tab^{t-1}, b^t)$

a. *(1 mark)* $(2, 3) \bigoplus (-2, 1) = (2(1) + 3(-2), (3)(1)) = (-4, 3)$

b. *(1 mark)* $-3 \bigodot (3, 1) = (-3(3)1^{-3-1}, 1^{-3}) = (-9, 1)$

c. *(3 marks)* Demonstrate whether the 5th axiom of vector spaces holds given that the $\vec{0}$ is $(0, 1)$. That is, do additive inverse exists for all vectors in V.

Let $\underline{v} = (v_1, v_2) \in V$ and let $\underline{w} = (w_1, w_2)$

$\underline{v} \oplus \underline{w} = \underline{0}$

$(v_1, v_2) \oplus (w_1, w_2) = (0, 1)$

$(v_1 w_2 + v_2 w_1, v_2 w_2) = (0, 1)$

① $v_1 w_2 + v_2 w_1 = 0$

② $v_2 w_2 = 1$

From ② $w_2 = \frac{1}{v_2}$ sub into ①

$v_1 \frac{1}{v_2} + v_2 w_1 = 0$

$v_2 w_1 = -\frac{v_1}{v_2}$

$w_1 = -\frac{v_1}{v_2^2}$

$\therefore \underline{w} = \left(\frac{-v_1}{v_2^2}, \frac{1}{v_2}\right) \in V$ since $\frac{-v_1}{v_2} \in \mathbb{R}$ and $\frac{1}{v_2} > 0$ since $v_2 > 0$.

Question 5.[3] *(4 marks)* Determine whether the set of all $n \times n$ matrices A such that $\text{trace}(A) = 0$ is a subspace of $\mathcal{M}_{n \times n}$.

Lets apply the subspace test and let $W = \{A \mid A \in M_{n\times n} \text{ and } tr(A) = 0\}$

① Closure under addition

Let $A, B \in W \Rightarrow tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = 0$

$tr(B) = b_{11} + b_{22} + \cdots + b_{nn} = 0$

$A + B \in W$ since
$$\begin{aligned}
tr(A+B) &= tr([a_{ij}] + [b_{ij}]) \\
&= tr([a_{ij} + b_{ij}]) \\
&= a_{11} + b_{11} + a_{22} + b_{22} + \cdots + a_{nn} + b_{nn} \\
&= a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} \\
&= tr(A) + tr(B) \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

② Closure under scalar multiplication

Let $A \in W \Rightarrow tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = 0$

and let $k \in \mathbb{R}$

$kA \in W$ since
$$\begin{aligned}
tr(kA) &= tr([ka_{ij}]) \\
&= ka_{11} + ka_{22} + \cdots + ka_{nn} \\
&= k(a_{11} + a_{22} + \cdots + a_{nn}) \\
&= k\,tr(A) \\
&= k(0) \\
&= 0
\end{aligned}$$

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[2] From http://www.math.uwaterloo.ca/ jmckinno/Math225/Week1/Lecture1e.pdf

[3] From the assigned homework.

Question 6.[3] *(4 marks)* Prove: If $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent and $\vec{v}_3$ does not lie in $\text{span}(\{\vec{v}_1, \vec{v}_2\})$, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Assume $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is linearly dependent. $\exists c_i$ s.t. $c_i \neq 0$ and $c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 = \underline{0}$

2 cases:

① $c_3 \neq 0$ then $c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 = \underline{0}$

$\underline{v}_3 = -\frac{c_2}{c_3}\underline{v}_2 - \frac{c_1}{c_3}\underline{v}_1 \in \text{span}(\{\underline{v}_1, \underline{v}_2\})$ ↯

③ $c_3 = 0$ then $c_1 \neq 0$ or $c_2 \neq 0$ and

$c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3 = \underline{0}$

$c_1\underline{v}_1 + c_2\underline{v}_2 + 0\,\underline{v}_3 = \underline{0}$

$c_1\underline{v}_1 + c_2\underline{v}_2 = \underline{0}$

a non-trivial linear combination that gives zero vector ↯

$\therefore$ $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is linearly independent.

Question 7. *(4 marks)* Determine whether $\{p_1(x) = 1 + 2x - 3x^2 + 5x^3, p_2(x) = 2 - x + 7x + 3x^2, p_3(x) = 3 - x + 3x^2 + 5x^3\}$ is linearly independent.

$$0 = c_1 p_1(x) + c_2 p_2(x) + c_3 p_3(x)$$

$$0 + 0x + 0x^2 + 0x^3 = c_1(1 + 2x - 3x^2 + 5x^3) + c_2(2 - x + 7x + 3x^2) + c_3(3 - x + 3x^2 + 5x^3)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ -3 & 7 & 3 \\ 5 & 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & -1 & 0 \\ -3 & 7 & 3 & 0 \\ 5 & 3 & 5 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \\ -2R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \\ -5R_1 + R_4 \to R_4 \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 13 & 12 & 0 \\ 0 & -7 & -10 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ \\ 5R_3 \to R_3 \\ 5R_4 \to R_4 \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 65 & 60 & 0 \\ 0 & -35 & -50 & \end{bmatrix}$$

$$\sim \begin{array}{c} \\ \\ 13R_2 + R_3 \to R_3 \\ -7R_2 + R_4 \to R_4 \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} 3R_4 + R_1 \to R_1 \\ -7R_4 + R_2 \to R_2 \\ -31R_4 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} -\frac{1}{5}R_2 \to R_2 \\ R_3 \leftrightarrow R_4 \end{array} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} -R_2 + R_1 \to R_1 \\ -R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ only trivial sol.

$\therefore$ set is linearly independent.

Question 8. *(4 marks)* In any vector space V, for any $\vec{u}, \vec{v}, \vec{w} \in V$ prove that if $\vec{v} + \vec{w} = \vec{u} + \vec{w}$ then $\vec{v} = \vec{u}$. *Show every step, justify every step, and cite the axiom(s) used!!!*

$\underline{v} + \underline{w} = \underline{u} + \underline{w}$

$\underline{v} + \underline{w} + \underline{w}' = \underline{u} + \underline{w} + \underline{w}'$ the additive inverse of $\underline{w}$ exists by axiom 5 (add $\underline{w}'$ on both sides)

$\underline{v} + (\underline{w} + \underline{w}') = \underline{u} + (\underline{w} + \underline{w}')$ associativity, axiom 3

$\underline{v} + \underline{0} = \underline{u} + \underline{0}$ axiom 5

$\underline{v} = \underline{u}$ axiom 4.

Bonus Question. *(1+5 marks)* State and prove the Exchange Lemma.