Dawson College: **Calculus II** (SCIENCE): 201-NYB-05-S3: Winter 2023: **Quiz 14** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \underbrace{\left(1+\frac{1}{n}\right)^{n^2}}_{a_n}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\left|\left(1+\frac{1}{n}\right)^{n^2}\right|\right)^{1/n}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} \quad \text{I.F. } 1^{\infty}$$

$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$$

$$\ln y = \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right)$$

$$\ln y = \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^{x}$$

$$\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \quad \text{I.F. } \infty\cdot 0$$

$$\ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad \text{I.F. } \frac{0}{0}$$

$$\ln y \overset{\hat{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}\cdot\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}}$$

$$\ln y = 1$$

$$y = e > 1 \quad \therefore \text{ diverges by the root test.}$$

Question 2. *(5 marks)* For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2}{(k(n+1))!}}{\frac{(n!)^2}{(kn)!}}\right|$$

$$= \lim_{n\to\infty} \frac{((n+1)!)^2}{(kn+k)!}\cdot\frac{(kn)!}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{(\cancel{n!}(n+1))^2}{\cancel{1\cdot 2\cdot 3\cdots kn}(kn+1)\cdots(kn+k)}\cdot\frac{\cancel{(kn)!}}{\cancel{(n!)^2}}$$

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{(kn+1)(kn+2)\cdots(kn+k)}$$

if $k=1$ then

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{1n+1} = \infty > 1 \quad \therefore \text{ diverges by ratio test}$$

if $k=2$ then

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{(2n+1)(2n+2)} = \frac{1}{4} < 1 \quad \therefore \text{ converges by ratio test}$$

if $k \geq 3$ then deg(denom.) > deg(num.)

$$= \lim_{n\to\infty} \frac{n^2+2n+1}{(kn+1)(kn+2)\cdots(kn+k)} = 0 < 1 \quad \therefore \text{ converges by ratio test.}$$

Question 3. *(5 marks)* Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

Lets determine if the series is abs. conv.

$$\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Let $f(x) = \frac{1}{x \ln x}$

- $f(x)$ is positive on $[2,\infty)$
- $f(x)$ is continuous on $(2,\infty)$
- $f'(x) = \frac{-1}{(x\ln x)^2}(\ln x + x\frac{1}{x}) < 0$ on $[2,\infty)$

$\therefore$ decreasing.

$$\int_2^\infty \frac{1}{x \ln x}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\ln x}\,dx$$

$u = \ln x$
$du = \frac{1}{x}dx$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u}\,du$$

$u(b) = \ln b$
$u(2) = \ln 2$

$$= \lim_{b\to\infty} \Big[\ln|u|\Big]_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty} \ln \underbrace{\ln b}_{\to\infty} - \ln\ln 2$$ diverges

$\therefore$ by integral test $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges

$\therefore$ not abs. conv.

$\sum_{n=2}^{\infty} (-1)^n b_n$ where $b_n = \frac{1}{n \ln n}$

1) $b_{n+1} \le b_n$ by
2) $\lim_{n\to\infty} b_n = 0$

$\therefore$ conv. by alt. series test
$\therefore$ conditionally convergent.

Bonus Question.[1] *(3 marks)* The Cantor ternary set $\mathscr{C}$ is created by iteratively deleting the open interval middle third from a set of line segments. One starts by deleting the open middle third $\left(\frac{1}{3},\frac{2}{3}\right)$ from the interval $[0,1]$, leaving two line segments: $\left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$. Next, the open middle third of each of these remaining segments is deleted, leaving four line segments: $\left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right]$. The Cantor ternary set contains all points in the interval $[0,1]$ that are not deleted at any step in this ad infinitum. Compute the length of the deleted intervals.

[1] https://en.wikipedia.org/wiki/Cantor$_s$*et*