Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S1: Winter 2023: **Quiz 14** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(4 marks)* Determine whether $H = \{A \mid A \in \mathscr{M}_{n\times n}$ and $BAC = CAB\}$ where B and C are fixed $n\times n$ matrices, is a subspace of $\mathscr{M}_{n\times n}$

Lets apply the subspace test, note $O \in H$ since $BOC = O = COB$, $\therefore H \neq \{\}$

① closure under addition

Let $A_1, A_2 \in H \Rightarrow BA_1C = CA_1B$ and $BA_2C = CA_2B$

$A_1 + A_2 \in H$ since $A_1 + A_2 \in M_{n\times n}$ and
$$\begin{aligned} B(A_1+A_2)C &= (BA_1+BA_2)C \\ &= BA_1C + BA_2C \\ &= CA_1B + CA_2B \quad \text{since } A_i \in H \\ &= C(A_1B + A_2B) \\ &= C(A_1+A_2)B \end{aligned}$$

② Closure under scalar multiplication

Let $r \in \mathbb{R}$ and $A \in H \Rightarrow BAC = CAB$

$rA \in H$ since $rA \in M_{n\times n}$ and
$$\begin{aligned} B(rA)C &= rBAC \\ &= rCAB \quad \text{since } A \in H \\ &= C(rA)B \end{aligned}$$

Question 2. *(4 marks)* Let $\overbrace{\{A_1, A_2, \ldots, A_k\}}^{S}$ be linearly independent in $\mathscr{M}_{m\times n}$, and suppose that U and V are invertible matrices of size $m\times m$ and $n\times n$, respectively. Show that $\underbrace{\{UA_1V, UA_2V, \ldots, UA_kV\}}_{S'}$ is linearly independent.

$O = c_1UA_1V + c_2UA_2V + \cdots + c_kUA_kV$ is a linear combination that gives the zero vector

since U and V are invertible
$$\begin{aligned} U^{-1}OV^{-1} &= U^{-1}(c_1UA_1V + c_2UA_2V + \cdots + c_kUA_kV)V^{-1} \\ O &= c_1U^{-1}UA_1VV^{-1} + c_2U^{-1}UA_2VV^{-1} + \cdots + c_kU^{-1}UA_kVV^{-1} \\ O &= c_1IA_1I + c_2IA_2I + \cdots + c_kIA_kI \\ O &= c_1A_1 + c_2A_2 + \cdots + c_kA_k \end{aligned}$$
$\Rightarrow$ the only solution is the trivial one since S is linearly independent

$\therefore S'$ is linearly independent.

Question 3. *(4 marks)* Determine whether the following statements is true or false. If the statement is false, provide a counterexample. If the statement is true, provide a proof of the statement.
If $\mathbf{u}, \mathbf{v}, \mathbf{p}, \mathbf{q}$ be nonzero vectors in $\mathbb{R}^3$ and $\text{span}\{\mathbf{u},\mathbf{v}\} = \text{span}\{\mathbf{p},\mathbf{q}\}$ then $(\mathbf{u}\times\mathbf{v})\times(\mathbf{p}\times\mathbf{q}) = \mathbf{0}$.

True,

Case ①: $\underline{u}$ and $\underline{v}$ are multiples of each other then $\underline{u} = k\underline{v}$ for some $k \in \mathbb{R}$
$$\begin{aligned} (\underline{u}\times\underline{v})\times(\underline{p}\times\underline{q}) &= (k\underline{v}\times\underline{v})\times(\underline{p}\times\underline{q}) \\ &= k(\underline{v}\times\underline{v})\times(\underline{p}\times\underline{q}) \\ &= k((\underline{0})\times(\underline{p}\times\underline{q})) \\ &= k\underline{0} \\ &= \underline{0} \end{aligned}$$

Case ②: $\underline{u}$ and $\underline{v}$ are not multiples of each other then $\underline{u}$ and $\underline{v}$ spans a plane through the origin and $\underline{p}$ and $\underline{q}$ also span the same plane.

$\therefore \underline{n}_1 = \underline{u}\times\underline{v}$ and $\underline{n}_2 = \underline{p}\times\underline{q}$ are both normals of the plane

$\therefore \underline{n}_1$ is parallel to $\underline{n}_2$ $\therefore \exists k$ s.t. $\underline{n}_1 = k\underline{n}_2$
$$\begin{aligned} (\underline{u}\times\underline{v})\times(\underline{p}\times\underline{q}) &= \underline{n}_1\times\underline{n}_2 \\ &= k\underline{n}_2\times\underline{n}_2 \\ &= k(\underline{n}_2\times\underline{n}_2) \\ &= k(\underline{0}) \\ &= \underline{0} \end{aligned}$$