Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S1: Winter 2023: **Quiz 15** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(2 marks)* Consider the vectors $\mathbf{u} = (1,2,-3)$ and $\mathbf{v} = (0,7,1)$ where $\mathscr{B} = \{\mathbf{u},\mathbf{v}\}$ is a basis for a vector space W. Find the coordinates of the vector $\mathbf{w} = (1,23,0)$ relative to $\mathscr{B}$.

$(\underline{w})_B = (c_1, c_2)$ where $\underline{w} = c_1\underline{u} + c_2\underline{v}$

$= (1,3)$

$(1,23,0) = c_1(1,2,-3) + c_2(0,7,1)$

$(1,23,0) = 1(1,2,-3) + 3(0,7,1)$

Question 2. *(5 marks)* Consider the space $M_{3\times 3}$ of all square matrices of size 3×3 and the subset W of skew-symmetric matrices:

$$W = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, \ a, b, c \text{ in } \mathbb{R} \right\}$$

and consider the matrices:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Determine whether $S = \{A,B,C\}$ is a basis for W and state the dimension of W.

① Does S span W? Let $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \in W$

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = c_1A + c_2B + c_3C$$

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = c_1\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$a = c_1$

$b = -c_1 + c_2 + c_3$

$c = c_3$

$$* \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\underline{c}} = \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\underline{b}}$$

since $\det A = 1 \neq 0$, the above is consistent $\forall a,b,c$ by the equiv. thm. $\therefore$ S spans W.

Is S linearly independent?

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = c_1A + c_2B + c_3C$$

By * and $a=b=c=0$ and the fact that $\det A \neq 0$ we only have the trivial solution $c_1 = c_2 = c_3 = 0$ by the equiv. thm.

$\therefore$ S is linearly independent.

$\therefore$ S is a basis

$\therefore$ $\dim(W) = 3$

Question 3. *(5 marks)* Let A be a nonzero 2×2 matrix and write $U = \{X \in \mathscr{M}_{2\times2} \mid XA = AX\}$. Show that $\dim(U) \geq 2$. *Hint: I and A are in U and prove by contradiction.*

Suppose $\dim(U) \leq 1$ $*$

Case ①: $A = kI$ where $k \in \mathbb{R}$

Then all $X \in M_{2\times2}$ commute with A. i.e. $XA = XkI$
$= kXI$
$= kIX$
$= AX$

$\therefore$ $U = M_{2\times2}$ and $\dim(U) = 4$ $\lightning$

Case ②: $\nexists k \in \mathbb{R}$ s.t. $A = kI$

$I \in U$ since $IA = AI$ $*$
$A \in U$ since $AA = AA$ $*$
since it is not the trivial subspace the basis β of U has at least one element. and by $*$ it follows that $\beta = \{B\}$.
$U = \operatorname{span}(\beta)$

$\left.\begin{array}{l} I = c_1 B \text{ by } * \\ A = c_1' B \text{ by } * \end{array}\right\} \Rightarrow \frac{1}{c_1} I = \frac{1}{c_1'} A$
$\frac{c_1'}{c_1} I = A$ $\lightning$

$\therefore$ $\dim(U) \geq 2$.

Bonus Question.
definition:[1] A *group* G is a set of elements satisfying the four conditions below, relative to some binary operation.

1. $\exists e \in G\, \forall g \in G : e\cdot g = g\cdot e = g$. (Identity.)
2. $\forall x,y,z \in G : (x\cdot y)\cdot z = x\cdot(y\cdot z)$. (Associativity.)
3. $\forall x \in G, \exists y \in G\; x\cdot y = y\cdot x = e$. (Inverse.)
4. $\forall x,y \in G : x\cdot y \in G$. (Closure.)

where e is called the *identity*.

a. *(2 marks)* Show that the identity is unique.
b. *(2 marks)* Is a vector space a group?

[1] https://web.williams.edu/Mathematics/sjmiller/public_html/mathlab/public_html/handouts/GroupTheoryIntro.tex