

Books, watches, notes or cell phones are not allowed. The only calculators allowed are the Sharp EL-531\*\*. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

**Question 1.** (1 mark each) Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. If  $A$  is a square matrix and  $\det(A + A^T) = 0$  then  $A$  might be singular.

**Question 2.** If  $A$  is an  $n \times n$  matrix, the characteristic polynomial  $c_A(x)$  of  $A$  is defined by  $c_A(x) = \det(xI - A)$ .

a. (3 marks) Show that if  $A$  is an  $n \times n$  matrix then  $c_{A^2}(x^2) = (-1)^n c_A(x) c_A(-x)$ .

$$\begin{aligned} c_{A^2}(x^2) &= \det(x^2 I - A^2) \\ &= \det((xI)^2 - A^2) \\ &= \det((xI - A)(xI + A)) \\ &= \det(-1(xI - A)(-xI - A)) \\ &= (-1)^n \det((xI - A)(-xI - A)) \\ &= (-1)^n \det(xI - A) \det(-xI - A) \\ &= (-1)^n c_A(x) c_A(-x) \end{aligned}$$

**Question 3.** (3 marks) Given  $\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$ ,  $B = \begin{bmatrix} 3g+a & 3h+b & 2 & 3i+c \\ d+2a & e+2b & 3 & f+2c \\ a & b & 4 & c \\ 0 & 0 & 5 & 0 \end{bmatrix}$  and  $\det B = 30$ . Find  $\det(5A^4(A^{-1})^T B^{-3} \text{adj}(A))$ .

$$\begin{aligned} &= 5^4 \det(A^4 (A^{-1})^T B^{-3} \text{adj}(A)) \\ &= 5^4 \det(A^4) \det(A^{-1})^T \det(B^{-3}) \det(\text{adj}(A)) \\ &= 5^4 (\det A)^4 \det A^{-1} (\det B)^{-3} (\det A)^{4-1} \\ &= 5^4 2^4 \frac{1}{\det A} \frac{1}{(\det B)^3} 2^3 \\ &= 5^4 2^7 \frac{1}{2} \frac{1}{(30)^3} = \frac{5^4 2^6}{(30)^3} \end{aligned}$$

**Question 3.** (3 marks) If  $A$  is an  $n \times n$  matrix where  $\det(A) = x \neq 0$  then determine for which value(s) of  $x$ , if any, the matrix  $A + \text{adj}(A^{-1})$  is invertible.

$$\begin{aligned} \text{For } \det(A + \text{adj}(A^{-1})) &= \det\left(A + \frac{1}{\det A} A\right) & \square^{-1} &= \frac{1}{\det \square} \text{adj} \square \\ &= \det\left(\left(1 + \frac{1}{\det A}\right) A\right) \\ &= \left(1 + \frac{1}{\det A}\right)^n \det A \neq 0 & \text{then } 1 + \frac{1}{\det A} &\neq 0 \\ & & \therefore \det A &\neq -1 \end{aligned}$$

**Question 4.** (1 mark) Correctly and precisely state Cramer's Rule.

Given the system  $A\underline{x} = \underline{b}$  where  $A$  is  $n \times n$ . If  $\det(A) \neq 0$  then the components of the unique solution of the system  $x_i = \frac{\det A_i}{\det A}$  where  $A_i$  is the matrix

$A$  with the  $i^{\text{th}}$  column of  $A$  replaced by  $\underline{b}$ .

**Bonus Question.** (5 marks) Prove Cramer's Rule.