Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S1: Winter 2023: **Quiz 8** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. If A is a square matrix and $\det(A+A^T)=0$ then A __might__ be singular.

Question 2. If A is an $n\times n$ matrix, the *characteristic polynomial* $c_A(x)$ of A is defined by $c_A(x)=\det(xI-A)$.

a. *(3 marks)* Show that if A is an $n\times n$ matrix then $c_{A^2}(x^2)=(-1)^n c_A(x)c_A(-x)$.

$$
\begin{aligned}
c_{A^2}(x^2) &= \det(x^2I-A^2)\\
&= \det((xI)^2-A^2)\\
&= \det((xI-A)(xI+A))\\
&= \det(-1(-xI+A)(xI+A))\\
&= (-1)^n \det((-xI+A)(xI+A))\\
&= (-1)^n \det(-xI+A)\det(xI+A)\\
&= (-1)^n c_A(-x)c_A(x)
\end{aligned}
$$

Question 3. *(3 marks)* Given $\det A = \begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = 2, B = \begin{bmatrix} 3g+a & 3h+b & 2 & 3i+c\\ d+2a & e+2b & 3 & f+2c\\ a & b & 4 & c\\ 0 & 0 & 5 & 0\end{bmatrix}$ and $\det B = 30$. Find $\det\left(5A^4(A^{-1})^TB^{-3}\mathrm{adj}(A)\right)$.

$$
\begin{aligned}
&= 5^4\det(A^4(A^{-1})^TB^{-3}\mathrm{adj}(A))\\
&= 5^4\det(A^4)\det(A^{-1})^T\det(B^{-3})\det(\mathrm{adj}(A))\\
&= 5^4(\det A)^4\det A^{-1}(\det B)^{-3}(\det A)^{4-1}\\
&= 5^4\, 2^4\,\frac{1}{\det A}\,\frac{1}{(\det B)^3}\,2^3\\
&= 5^4\, 2^7\,\frac{1}{2}\,\frac{1}{(30)^3} = \frac{5^4 2^6}{(30)^3}
\end{aligned}
$$

Question 3. *(3 marks)* If A is an an $n\times n$ matrix where $\det(A)=x\neq 0$ then determine for which value(s) of x, if any, the matrix $A+\mathrm{adj}(A^{-1})$ is invertible.

$$\square^{-1} = \frac{1}{\det\square}\,\mathrm{adj}\,\square$$

For
$$
\begin{aligned}
\det(A+\mathrm{adj}(A^{-1})) &= \det(A+\det A^{-1}(A^{-1})^{-1})\\
&= \det\left(A+\frac{1}{\det A}A\right)\\
&= \det\left(\left(1+\frac{1}{\det A}\right)A\right)\\
&= \left(1+\frac{1}{\det A}\right)^n\det A\neq 0 \quad \text{then} \quad 1+\frac{1}{\det A}\neq 0
\end{aligned}
$$

$\therefore \det A \neq -1$

Question 4. *(1 mark)* Correctly and precisely state Cramer's Rule.

Given the system $A\underline{x}=\underline{b}$ where A is $n\times n$. If $\det(A)\neq 0$ then the components of the unique solution of the system $x_i = \dfrac{\det A_i}{\det A}$ where A_i is the matrix A with the i^{th} column of A replaced by $\underline{b}$.

Bonus Question. *(5 marks)* Prove Cramer's Rule.