Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S1: Winter 2024: **Quiz 12** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks)* Determine whether the set V of all functions $f : \mathbb{R} \to \mathbb{R}$ with addition $(f+g)(x) = f(x) + g(x)$ and scalar multiplication $(af)(x) = f(ax)$ is a vector space.

$V = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$, V is not a vector space since axiom 7 fails,

that is $(r+s)\underline{u} = r\underline{u} + s\underline{u}$

Let $f(x) = x^2 \in V$ and $r = s = 1$

$$\begin{aligned} LHS &= (1+1) \cdot_V x^2 \\ &= 2 \cdot_V x^2 \\ &= (2x)^2 \\ &= 4x^2 \end{aligned}$$

$$\begin{aligned} RHS &= 1 \cdot_V x^2 + 1 \cdot_V x^2 \\ &= (1 \cdot x)^2 + (1 \cdot x)^2 \\ &= x^2 + x^2 \\ &= 2x^2 \neq LHS \end{aligned}$$

Question 2. *(3 marks)* Show that the $\mathbf{0}$ in any vector space is unique. *Show every step, justify every step, and cite the axiom(s) used!!!*

Suppose the $\underline{0}$ is not unique. That is $\exists\ \underline{z}_1$ and $\underline{z}_2$ where $\underline{z}_1 \neq \underline{z}_2$, $\underline{z}_i \in V$

and $\underline{v} + \underline{z}_i = \underline{v} \quad \forall \underline{v} \in V$

we then have $\underline{v} + \underline{z}_1 = \underline{v} + \underline{z}_2$

by axiom 5 $\exists \underline{w} \in W$ s.t. $\underline{w} + \underline{v} = \underline{z}_1$

$$\begin{aligned} \underline{w} + (\underline{v} + \underline{z}_1) &= \underline{w} + (\underline{v} + \underline{z}_2) \\ (\underline{w} + \underline{v}) + \underline{z}_1 &= (\underline{w} + \underline{v}) + \underline{z}_2 && \text{by axiom 3} \\ \underline{z}_1 + \underline{z}_1 &= \underline{z}_1 + \underline{z}_2 && \text{by axiom 5} \\ \underline{z}_1 &= \underline{z}_2 && \text{by axiom 4} \end{aligned}$$

$\therefore$ $\underline{0}$ is unique.

Question 3.[1] Let $V = \{(a,b) \mid a, b \in \mathbb{R}, b > 0\}$. And the addition in V is defined by $(a,b) \bigoplus (c,d) = (ad + bc, bd)$ and scalar multiplication in V is defined by $t \bigodot (a,b) = (tab^{t-1}, b^t)$

a. *(2 marks)* If V is a vector space find the zero vector of the vector space.

Since V is a vector space and by thm 1.1 we have that

$$\underline{0} = 0 \odot \underline{v} = 0 \odot (a,b) = (0ab^{-1}, b^0) = (0,1)$$

b. *(3 marks)* Demonstrate whether the 3rd axiom of vector spaces holds. That is, vector addition is associative.

$\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$

Let $\underline{u} = (u_1, u_2)$, $\underline{v} = (v_1, v_2)$, $\underline{w} = (w_1, w_2)$ $\in V$

$$\begin{aligned} LHS &= \underline{u} \oplus (\underline{v} \oplus \underline{w}) \\ &= (u_1, u_2) \oplus ((v_1, v_2) \oplus (w_1, w_2)) \\ &= (u_1, u_2) \oplus (v_1 w_2 + v_2 w_1, v_2 w_2) \\ &= (u_1 v_2 w_2 + u_2(v_1 w_2 + v_2 w_1), u_2 v_2 w_2) \\ &= (u_1 v_2 w_2 + u_2 v_1 w_2 + u_2 v_2 w_1, u_2 v_2 w_2) \end{aligned}$$

$$\begin{aligned} RHS &= (\underline{u} \oplus \underline{v}) \oplus \underline{w} \\ &= ((u_1, u_2) \oplus (v_1, v_2)) \oplus (w_1, w_2) \\ &= (u_1 v_2 + u_2 v_1, u_2 v_2) \oplus (w_1, w_2) \\ &= ((u_1 v_2 + u_2 v_1) w_2 + w_1 u_2 v_2, u_2 v_2 w_2) \\ &= (u_1 v_2 w_2 + u_2 v_1 w_2 + w_1 u_2 v_2, u_2 v_2 w_2) \\ &= LHS \end{aligned}$$

[1]From http://www.math.uwaterloo.ca/ jmckinno/Math225/Week1/Lecture1e.pdf