Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S1: Winter 2024: **Quiz 7** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(1 mark each)* Complete each of the following sentences with MUST, MIGHT, or CANNOT.

a. If A is a 3×3 matrix and B is obtained from A by multiplying the first column by 4 and multiplying the third column by $\frac{3}{4}$, then $\det(B)$ __must__ be equal to $3\det(A)$.

Question 1. *(4 marks)* Given the polynomial $p(x) = a + bx + cx^2 + dx^3 + x^4$, the matrix $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -b & -c & -d \end{bmatrix}$ is called the companion matrix of $p(x)$. Show that $\det(xI - C) = p(x)$

$$\det(xI-C) = \begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a & b & c & x+d \end{vmatrix} = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41}$$

$$= x\begin{vmatrix} x & -1 & 0 \\ 0 & x & -1 \\ b & c & x+d \end{vmatrix} - a\begin{vmatrix} -1 & 0 & 0 \\ x & -1 & 0 \\ 0 & x & -1 \end{vmatrix}$$

$$= x\left[x\begin{vmatrix} x & -1 \\ c & x+d \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ b & x+d \end{vmatrix}\right] - a(-1)(-1)(-1)$$

$$= x\left[x(x(x+d)+c) + b\right] + a$$

$$= x^4 + dx^3 + cx^2 + bx + a$$

Question 2. *(3 marks)* If A is an $n\times n$ matrix, the *characteristic polynomial* $c_A(x)$ of A is defined by $c_A(x) = \det(xI - A)$. The *Cayley-Hamilton Theorem* states that for any square matrix A, $c_A(x) = 0$ when evaluated at $x = A$. Prove the *Cayley-Hamilton Theorem* for 2×2 matrices. **Important Hint: First find the characteristic polynomial!**

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $c_A(x) = \det(xI - A) = \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = (x-a)(x-d) - bc$

$= x^2 - (a+d)x + ad - bc$

$$c_A(A) = A^2 - (a+d)A + (ad-bc)I$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a+d)A + (ad-bc)I$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix} - \begin{bmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Question 3. *(5 marks)* Only using elementary operations show that

$$\underbrace{\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}_{A} = (1-t^2)\underbrace{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}_{B}$$

Case 1: $t=1$ or $t=-1$ then RHS $=0$ and LHS $=0$ since R_1 and R_2 are multiples of each other.

Case 2: $t\neq 1$ and $t\neq -1$

$$A \overset{-tR_2+R_1\to R_1}{=} \begin{vmatrix} a_1-a_1t^2 & a_2-a_2t^2 & a_3-a_3t^2 \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\overset{\frac{1}{1-t^2}R_1\to R_1}{=} (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\overset{-tR_1+R_2\to R_2}{=} (1-t^2)\,B$$