Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S8: Winter 2024: **Quiz 13** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks)* Determine whether the set $V = \mathscr{M}_{2\times 2}$ of all 2×2 matrices with the usual addition and scalar multiplication defined as $a \cdot X = aX^T$ is a vector space.

V with the given operations is not a vector space since axiom 10 fails

Let $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in V$ but $1 \cdot X = 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq X$

Question 2. *(3 marks)* Show that $\mathbf{x} = \mathbf{v}$ is the only solution of the equation $\mathbf{x} + \mathbf{x} = 2\mathbf{v}$ in a vector space V. *Show every step, justify every step, and cite the axiom(s) used!!!*

$\underline{x} + \underline{x} = 2\underline{v}$

$1\cdot\underline{x} + 1\cdot\underline{x} = 2\underline{v}$ axiom 10

$(1+1)\cdot\underline{x} = 2\underline{v}$ axiom 7

$2\cdot\underline{x} = 2\underline{v}$

$\frac{1}{2}(2\cdot\underline{x}) = \frac{1}{2}(2\underline{v})$

$(\frac{1}{2}2)\underline{x} = (\frac{1}{2}2)\underline{v}$ axiom 9

$1\cdot\underline{x} = 1\cdot\underline{v}$

$\underline{x} = \underline{v}$ axiom 10

Question 3.[1] Let $V = \{(a,b) \mid a,b \in \mathbb{R}, b > 0\}$. And the addition in V is defined by $(a,b) \bigoplus (c,d) = (ad+bc, bd)$ and scalar multiplication in V is defined by $t \bigodot (a,b) = (tab^{t-1}, b^t)$

a. *(2 marks)* If V is a vector space and $\mathbf{v} \in V$ find the additive inverse of $\mathbf{v}$.

Since V is a vector space and by thm 1.1 the additive inverse of $\underline{u} = (a,b) \in V$ is $(-1)\bigodot(a,b) = (-ab^{-2}, b^{-1})$.

b. *(3 marks)* Demonstrate whether the 7th axiom of vector spaces holds. That is, addition of scalars distributes over scalar multiplication, $(r+s)\bigodot\mathbf{v} = r\bigodot\mathbf{v} \bigoplus s\bigodot\mathbf{v}$.

LHS $= (r+s)\bigodot(a,b)$
$= ((r+s)ab^{r+s-1}, b^{r+s})$

RHS $= r\bigodot(a,b) \bigoplus s\bigodot(a,b)$
$= (rab^{r-1}, b^r) \bigoplus (sab^{s-1}, b^s)$
$= (rab^{r-1}b^s + sab^{s-1}b^r, b^r b^s)$
$= (rab^{s+r-1} + sab^{s+r-1}, b^{r+s})$
$= ((r+s)ab^{s+r-1}, b^{r+s})$
$=$ LHS.

[1]From http://www.math.uwaterloo.ca/ jmckinno/Math225/Week1/Lecture1e.pdf

Bonus Question.

definition:[2] A *group* G is a set of elements satisfying the four conditions below, relative to some binary operation.

1. $\exists e \in G$ such that $\forall g \in G : e \cdot g = g \cdot e = g$. (Identity.)
2. $\forall x, y, z \in G : (x \cdot y) \cdot z = x \cdot (y \cdot z)$. (Associativity.)
3. $\forall x \in G, \exists y \in G$ such that $x \cdot y = y \cdot x = e$. (Inverse.)
4. $\forall x, y \in G : x \cdot y \in G$. (Closure.)

where e is called the *identity*.

a. *(2 marks)* Show that the identity is unique.

b. *(2 marks)* Is a vector space a group?

[2] https://web.williams.edu/Mathematics/sjmiller/public_html/mathlab/public_html/handouts/GroupTheoryIntro.tex