Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S1: Winter 2025: **Quiz 2** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks each)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. If A is an $m \times n$ matrix then $\operatorname{tr}(A^TA) \geq 0$.

True, Let $A = [a_{ij}]_{m \times n}$

$$A^TA = \left[\sum_{k=1}^{m} a_{ki}a_{kj}\right]$$

$$\operatorname{tr}(A^TA) = \sum_{k=1}^{m} a_{k1}a_{k1} + \sum_{k=1}^{m} a_{k2}a_{k2} + \dots + \sum_{k=1}^{m} a_{kn}a_{kn}$$

$$= \sum_{l=1}^{n}\sum_{k=1}^{m} a_{kl}^2 \geq 0$$

Question 3. *(5 marks)* Find the reduced row echelon form of $\begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix}$ where $c \neq a$ and $b \neq a$.

$$\sim \begin{matrix} -R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a & b+c \\ 0 & b-a & -(b-a) \\ 0 & c-a & -(c-a) \end{bmatrix}$$

$$\sim \begin{matrix} \frac{1}{b-a}R_2 \to R_2 \\ \frac{1}{c-a}R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & a & b+c \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$ since $b \neq a$ and $c \neq a$

$$= \begin{matrix} -aR_2+R_1 \to R_1 \\ -R_2+R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 0 & a+b+c \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 2. *(3 marks)* Find all 2×2 matrices M such that $MA - AM = 0$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$0 = MA - AM$

$$0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$0 = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} - \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

$$0 = \begin{bmatrix} -c & a-d \\ 0 & c \end{bmatrix}$$

$\Rightarrow c = 0$

$b = s \in \mathbb{R}$

$d = t \in \mathbb{R}$

$\to a = d = t$

$\therefore (a,b,c,d) = (t, s, 0, t) \quad s, t \in \mathbb{R}$

Question 3.[1] *(6 marks)* Find the values, if any, of h and k for which the following system has:

$$\begin{cases} x + 3y + 2z = k+5 \\ -x + (h-1)y + (h^2-6)z = k-1 \\ 3x + 9y + (h^2-h)z = k^2+3k+11 \end{cases}$$

Exactly one solution, no solutions, infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & k+5 \\ -1 & h-1 & h^2-6 & k-1 \\ 3 & 9 & h^2-h & k^2+3k+11 \end{array}\right]$$

$$\begin{array}{l} \sim R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3 \end{array} \left[\begin{array}{cccc} 1 & 3 & 2 & k+5 \\ 0 & h+2 & h^2-4 & 2k+4 \\ 0 & 0 & h^2-h-6 & k^2-4 \end{array}\right]$$

$$= \left[\begin{array}{cccc} 1 & 2 & 2 & k+5 \\ 0 & h+2 & (h+2)(h-2) & 2(k+2) \\ 0 & 0 & (h-3)(h+2) & (k-2)(k+2) \end{array}\right]$$

If $h=3$ then $\left[\begin{array}{cccc} 1 & 2 & 2 & k+5 \\ 0 & 5 & 5 & 2(k+2) \\ 0 & 0 & 0 & (k-2)(k+2) \end{array}\right]$

and $k=-2$ or 2 then the system has ∞-many sol. since #leading entries $<$ #var.

and $k\neq -2$ or 2 then the system has no solutions since there is a leading entry in the constant column.

If $h=-2$ then $\left[\begin{array}{cccc} 1 & 2 & 2 & k+5 \\ 0 & 0 & 0 & 2(k+2) \\ 0 & 0 & 0 & (k-2)(k+2) \end{array}\right]$

and $k=-2$ then the system has ∞-many sol. since #leading entries $<$ #var.

and $k\neq -2$ then the system has no solutions since there is a leading entry in the constant column.

If $h\neq -2, 3$ then # leading entry = # var, therefore the system has a unique solution.

Bonus Question. *(3 marks)* If A, B and C are matrices such that the operations are defined, show that $A(BC) = (AB)C$.

[1]From a John Abbott Final Examination