Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S1: Winter 2025: **Quiz 3** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks each)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. If A is row equivalent to a product of elementary matrices, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$.

True,

$A \sim$ k elementary row operation $\sim E \sim$ inverse elementary row operation from which E was obtained $\sim I$

$\therefore$ the RREF of A is I $\therefore$ by the equivalence theorem $A\underline{x} = \underline{b}$ has a unique solution $\forall \underline{b}$

Question 2. *(5 marks)* Solve for the matrix X in the following equation:

$$\left(\begin{bmatrix}1 & 3\\0 & 2\end{bmatrix} X^{-1} + I\right)^T = \left(\begin{bmatrix}-1 & 4\\1 & -3\end{bmatrix} X^T\right)^{-1}$$

$$\left(\begin{bmatrix}1 & 3\\0 & 2\end{bmatrix} X^{-1}\right)^T + I^T = (X^T)^{-1} \begin{bmatrix}-1 & 4\\1 & -3\end{bmatrix}^{-1}$$

$$(X^{-1})^T \begin{bmatrix}1 & 3\\0 & 2\end{bmatrix}^T + I = (X^{-1})^T \frac{1}{3-4} \begin{bmatrix}-3 & -4\\-1 & -1\end{bmatrix}$$

$$I = (X^{-1})^T \begin{bmatrix}3 & 4\\1 & 1\end{bmatrix} - (X^{-1})^T \begin{bmatrix}1 & 0\\3 & 2\end{bmatrix}$$

$$I = (X^{-1})^T \left(\begin{bmatrix}3 & 4\\1 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\3 & 2\end{bmatrix}\right)$$

$$I = (X^{-1})^T \begin{bmatrix}2 & 4\\-2 & -1\end{bmatrix}$$

$$I = (X^T)^{-1} \begin{bmatrix}2 & 4\\-2 & -1\end{bmatrix}$$

$$X^T I = X^T (X^T)^{-1} \begin{bmatrix}2 & 4\\-2 & -1\end{bmatrix}$$

$$X^T = \begin{bmatrix}2 & 4\\-2 & -1\end{bmatrix}$$

$$X = \begin{bmatrix}2 & -2\\4 & -1\end{bmatrix}$$

Question 3. *(3 marks)* Let A be an $n \times n$ matrix such that $A^n = nA - (n-1)I$ where $n \neq 1$. Show that A is invertible, and find A^{-1} in terms of A and I_n.

$$(n-1)I = nA - A^n$$

$$(n-1)I = A(nI - A^{n-1})$$

$$I = A\frac{1}{n-1}(nI - A^{n-1})$$

$\therefore$ A is invertible and $A^{-1} = \frac{1}{n-1}(nI - A^{n-1})$.

Question 4. *(5 marks)* Express

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim_{\substack{2R_3 + R_1 \to R_1 \\ \frac{1}{2}R_2 \to R_2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since the RREF of A is I, A is invertible by equivalence theorem

and A^{-1} as a product of elementary matrices.

$$E_3E_2E_1A = I$$

$$E_3E_2E_1AA^{-1} = IA^{-1}$$

$$A^{-1} = E_3E_2E_1$$

$$I_3 \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$I_3 \sim_{2R_3 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

$$I_3 \sim_{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

$$I_3 \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1^{-1}$$

$$I_3 \sim_{-2R_3 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2^{-1}$$

$$I_3 \sim_{2R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3^{-1}$$

$$A^{-1} = E_3E_2E_1$$

$$(A^{-1})^{-1} = (E_3E_2E_1)^{-1}$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}$$