Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S1: Winter 2025: **Quiz 4** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks each)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. If A is any skew-symmetric matrix then A^2 is skew-symmetric matrix.

False, Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, A is skew-symmetric since $A^T = -A$

But $A^2 = AA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, A^2 is symmetric since $(A^2)^T = A^2$.

Question 2. *(5 marks)* Let A and B be two 3×3 matrices such that $\det(A) = \begin{vmatrix} a+2b & c & b \\ d+2e & f & e \\ g+2h & i & h \end{vmatrix} = -2$ where $a,d,g \neq 0$, and $\det(B) = 3$. Find the following: $adg \begin{vmatrix} 2 & 2 & 2 \\ \frac{b}{a} & \frac{e}{d} & \frac{h}{g} \\ \frac{c}{a} & \frac{f}{d} & \frac{i}{g} \end{vmatrix}$.

$-2C_3 + C_1 \to C_1$

$\begin{vmatrix} a & c & b \\ d & f & e \\ g & i & h \end{vmatrix} = -2$

$C_1 \leftrightarrow C_2$

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$

$aC_1 \to C_1 \quad dC_2 \to C_2 \quad gC_3 \to C_3$

$= adg \frac{1}{a}\frac{1}{d}\frac{1}{g} \begin{vmatrix} 2a & 2d & 2g \\ b & e & h \\ c & f & i \end{vmatrix}$

$\frac{1}{2}R_1 \to R_1$

$= 2 \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

$= 2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(2) = 4$ since $\det(A^T) = \det(A)$

Question 3. *(3 marks)* Prove that if $A^T A = A$, then A is symmetric and $A = A^2$.

Premise:

$A^T A = A$

conclusion:

A is symmetric

$A = A^2$

$A^T = (A^T A)^T$ by premise

$= A^T (A^T)^T$

$= A^T A$

$= A$

$\therefore A$ is symmetric

$A^2 = AA$

$= A^T A$ since A is symmetric

$= A$.

Question 4. *(5 marks)* If A is an $n \times n$ matrix, the *characteristic polynomial* $c_A(x)$ of A is defined by $c_A(x) = \det(xI - A)$.

a. *(5 marks)* Find the eigenvalues λ of $A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$. That is, find the values of λ for which $c_A(\lambda) = 0$.

b. *(3 marks bonus)* Find the eigenvector for a non-zero eigenvalue found in part a. That is, find a non-trivial solution $\mathbf{x}$ for each λ where $A\mathbf{x} = \lambda\mathbf{x}$.

$$0 = \det\left(\lambda I - \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}\right)$$

$$= \begin{vmatrix} \lambda-2 & -2 & 2 \\ -1 & \lambda-3 & 1 \\ 1 & -1 & \lambda-1 \end{vmatrix}$$

$$= \underset{C_3+C_2 \to C_2}{} \begin{vmatrix} \lambda-2 & 0 & 2 \\ -1 & \lambda-2 & 1 \\ 1 & \lambda-2 & \lambda-1 \end{vmatrix}$$

$$\underset{R_2+R_3\to R_3}{=} \begin{vmatrix} \lambda-2 & 0 & 2 \\ -1 & \lambda-2 & 1 \\ 0 & 2\lambda-4 & \lambda \end{vmatrix}$$

$$= (\lambda-2)\begin{vmatrix} \lambda-2 & 1 \\ 2\lambda-4 & \lambda \end{vmatrix} + 2\begin{vmatrix} -1 & \lambda-2 \\ 0 & 2\lambda-4 \end{vmatrix}$$

$$= (\lambda-2)\left[(\lambda-2)\lambda - (2\lambda-4)\right] - 2\left[2\lambda-4\right]$$

$$= (\lambda-2)\left[\lambda^2-2\lambda-2\lambda+4\right] - 4\left[\lambda-2\right]$$

$$= (\lambda-2)\left[\lambda^2-4\lambda\right]$$

$$= (\lambda-2)\lambda(\lambda-4)$$

$\lambda = 2$ $\quad \lambda = 0$ $\quad \lambda = 4$

b) $A\underline{x} = \lambda\underline{x}$

$A\underline{x} - \lambda\underline{x} = \underline{0}$

$(A - \lambda I)\underline{x} = \underline{0}$

$\lambda = 2$: $\begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$

$\underset{R_1+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix}$

$\underset{\substack{\frac{1}{2}R_2\to R_2 \\ -R_2+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\underset{-R_2+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\underset{-R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Let $z = t \quad t \in \mathbb{R}$

$x = 0$

$y = t$

$\therefore \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = 2$.