Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S1: Winter 2025: **Quiz 5** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks each)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. If A and B are $n \times n$ matrices and $BA^2 + B^2A$ is invertible then $A + B$ is invertible.

True,

since BA^2+B^2A is invertible $\det(BA^2+B^2A) \neq 0$

$\det(B(A+B)A) \neq 0$

$\det(B)\det(A+B)\det B \neq 0$

$\therefore \det(A+B) \neq 0$

$\therefore A+B$ is invertible.

Question 2. *(3 marks)* Let A denote an invertible $n \times n$ matrix where $n \geq 2$. Show that $\operatorname{adj}(\operatorname{adj}(\mathrm{A})) = (\det A)^{n-2}A$.

$\square^{-1} = \frac{1}{\det \square} \operatorname{adj} \square$

$\operatorname{adj} \square = (\det \square)\square^{-1}$

$\operatorname{adj} A = (\det A) A^{-1}$

$LHS = \operatorname{adj}((\det A)A^{-1})$

$= \det((\det A)A^{-1})((\det A)(A^{-1}))^{-1}$

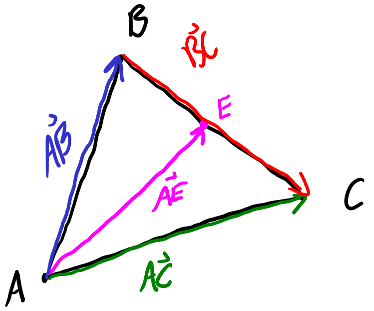
$= (\det A)^n \det A^{-1} \frac{1}{\det A}(A^{-1})^{-1}$

$= (\det A)^n \frac{1}{\det A}\frac{1}{\det A} A$

$= (\det A)^{n-2} A$

$= RHS$

Question 3. *(5 marks)* Let A, B, and C denote the three vertices of a triangle. If E is the midpoint of side BC, show that: $\vec{AE} = \frac{1}{2}(\vec{AB} + \vec{AC})$.



We have that $\vec{AC} = \vec{AB} + \vec{BC}$

$2\vec{BE} = \vec{BC}$

$$\begin{aligned} RHS &= \frac{1}{2}(\vec{AB} + \vec{AC}) \\ &= \frac{1}{2}(\vec{AB} + \vec{AB} + \vec{BC}) \\ &= \frac{1}{2}(2\vec{AB} + 2\vec{BE}) \\ &= \vec{AB} + \vec{BE} \\ &= \vec{AE} \end{aligned}$$

Question 4.[1] Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Given: $||\vec{u}|| = 5$, $||\vec{u} + 2\vec{v}|| = \sqrt{2}$, $\vec{v}$ and $\vec{u} + 3\vec{v}$ are both unit vectors, and the angle between $\vec{u} + 2\vec{v}$ and $\vec{u} + 3\vec{v}$ is $\pi/4$.

a. *(3 marks)* Find $\vec{u} \cdot \vec{v}$.

b. *(2 marks)* Find $||\vec{u} + \vec{v}||$.

a)

$$(\underline{u} + 2\underline{v}) \cdot (\underline{u} + 3\underline{v}) = ||\underline{u} + 2\underline{v}|| \, ||\underline{u} + 3\underline{v}|| \cos\frac{\pi}{4}$$

$$\underline{u} \cdot \underline{u} + \underline{u} \cdot (3\underline{v}) + (2\underline{v}) \cdot \underline{u} + (2\underline{v}) \cdot (3\underline{v}) = \sqrt{2} \cdot 1 \, \frac{1}{\sqrt{2}}$$

$$||\underline{u}||^2 + 3\underline{u} \cdot \underline{v} + 2\underline{v} \cdot \underline{u} + 6\underline{v} \cdot \underline{v} = 1$$

$$5^2 + 5\underline{u} \cdot \underline{v} + 6||\underline{v}||^2 = 1$$

$$\begin{aligned} 5\underline{u} \cdot \underline{v} &= -24 - 6(1)^2 \\ 5\underline{u} \cdot \underline{v} &= -30 \\ \underline{u} \cdot \underline{v} &= -6 \end{aligned}$$

b)

$$\begin{aligned} ||\underline{u} + \underline{v}||^2 &= (\underline{u} + \underline{v}) \cdot (\underline{u} + \underline{v}) \\ &= \underline{u} \cdot \underline{u} + \underline{v} \cdot \underline{u} + \underline{u} \cdot \underline{v} + \underline{v} \cdot \underline{v} \\ &= ||\underline{u}||^2 + 2\underline{u} \cdot \underline{v} + ||\underline{v}||^2 \\ &= 5^2 + 2(-6) + 1^2 \\ &= 25 - 12 + 1 \\ &= 14 \end{aligned}$$

$$||\underline{u} + \underline{v}|| = \sqrt{14}$$

[1]From or modified from a John Abbott final examination