Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S1: Winter 2025: **Quiz 8** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(5 marks)* Let V be the solution space of the equation $4x - y + 2z = 0$, and let W be the subspace of $\mathbb{R}^3$ spanned by $(1, 1, 1)$. Find a vector $\mathbf{v}$ in V and a vector $\mathbf{w}$ in W for which $\mathbf{v} + \mathbf{w} = (1, 0, 1)$

Let $y = s$, $z = t$ $\quad s,t \in \mathbb{R}$

$4x - s + 2t = 0$

$4x = s - 2t$

$x = \frac{1}{4}s - \frac{1}{2}t$

$\therefore \underline{x} = (x,y,z) = \left(\frac{1}{4}s - \frac{1}{2}t, s, t\right)$

$= s\underbrace{\left(\frac{1}{4}, 1, 0\right)}_{\underline{d_1}} + t\underbrace{\left(-\frac{1}{2}, 0, 1\right)}_{\underline{d_2}}$

$V = \text{span}(\{\underline{d_1}, \underline{d_2}\})$

$W = \text{span}(\{(1,1,1)\})$

$\underline{v} \in V \Rightarrow \underline{v} = c_1\underline{d_1} + c_2\underline{d_2}$

$\underline{w} \in W \Rightarrow \underline{w} = c_3(1,1,1)$

$\underline{v} + \underline{w} = (1,0,1)$

$c_1\left(\frac{1}{4}, 1, 0\right) + c_2\left(-\frac{1}{2}, 0, 1\right) + c_3(1,1,1) = (1,0,1)$

$$\left[\begin{array}{ccc|c} \frac{1}{4} & -\frac{1}{2} & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim_{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\sim_{-\frac{1}{4}R_1 + R_2 \to R_2} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{3}{4} & 1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\sim_{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{4} & 1 \end{array}\right]$$

$$\sim_{\frac{1}{2}R_2 + R_3 \to R_3} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \frac{5}{4} & \frac{3}{2} \end{array}\right]$$

$$\sim_{\frac{4}{5}R_3 \to R_3} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{6}{5} \end{array}\right]$$

$$\sim_{\substack{-R_3 + R_1 \to R_1 \\ -R_3 + R_2 \to R_2}} \left[\begin{array}{cccc} 1 & 0 & 0 & -\frac{6}{5} \\ 0 & 1 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{6}{5} \end{array}\right]$$

$\underline{v} = -\frac{6}{5}\left(\frac{1}{4}, 1, 0\right) - \frac{1}{5}\left(-\frac{1}{2}, 0, 1\right)$

$\underline{w} = \frac{6}{5}(1,1,1)$

Question 2. *(5 marks)* If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly dependent nonzero vectors, then at least one vector $\mathbf{v}_k$ is a unique linear combination of $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$.

Let $S_i = \{\underline{v}_1, \ldots, \underline{v}_i\}$ where $1 \le i \le n$

$S_1 = \{\underline{v}_1\}$ is lin. ind since $\underline{v}_i \neq \underline{0}$ $\forall i$

$S_2 = \{\underline{v}_1, \underline{v}_2\}$ is either lin. ind. or lin. dep. if lin. ind. we consider the next set S_3 (and so on). if S_2 is lin. dep then $\underline{v}_1$ is a multiple $\underline{v}_2$

$\vdots$

Suppose $S_{i-1} = \{\underline{v}_1, \ldots, \underline{v}_{i-1}\}$ is lin. ind. (then no vector in that set can be written as a lin. comb. of the others) and $\underline{S}_i$ is lin. dep. then $\underline{v}_i$ can be written as a lin. comb. of S_{i-1}.

Suppose there exists two ways to express $\underline{v}_i$ as a lin. comb.

① $\underline{S_i} = c_1\underline{v}_1 + \cdots + c_k\underline{v}_k + \cdots + c_{i-1}\underline{v}_{i-1}$

② $\underline{S_i} = d_1\underline{v}_1 + \cdots + d_k\underline{v}_k + \cdots + d_{i-1}\underline{v}_{i-1}$

where $c_k \neq d_k$

①-② $\underline{0} = (c_1 - d_1)\underline{v}_1 + \ldots + (c_k - d_k)\underline{v}_k + \ldots + (c_{i-1} - d_{i-1})\underline{v}_{i-1}$

non trivial solution to above since $c_k - d_k \neq 0$

$\therefore S_{i-1}$ is lin. dep $\lightning$ $\therefore$ Can be written in a unique way.

Question 3. Let W be the subspace of all polyomials in $\mathbb{P}_3$ such that $p(1) = 0$

a. *(4 marks)* Find a basis $\mathscr{B}$ of W.

b. *(1.5 marks)* State $\dim(\mathbb{P}_3)$, $\dim(W)$, and $\dim(\{0+0x+0x^2+0x^3\})$.

c. *(1 mark)* Find the coordinate vector of $p(x) = 1+x+x^2-3x^3$ relative to the basis found in part a.

Let $p(x) = a+bx+cx^2+dx^3 \in P_3$

$$0 = p(1)$$
$$0 = a+b+c+d$$
$$a = -b-c-d$$

$\therefore$ $p(x) = (-b-c-d) + bx + cx^2 + dx^3 \in W$

$$= b\underbrace{(-1+x)}_{p_1(x)} + c\underbrace{(-1+x^2)}_{p_2(x)} + d\underbrace{(-1+x^3)}_{p_3(x)}$$

$\therefore$ $\beta = \{p_1(x), p_2(x), p_3(x)\}$ spans W

Linear independence:

$$\underline{0} = c_1(-1+x) + c_2(-1+x^2) + c_3(-1+x^3)$$
$$\underline{0} = (-c_1-c_2-c_3)(1) + c_1(x) + c_2(x^2) + c_3(x^3)$$

$\Rightarrow c_1 = c_2 = c_3 = 0$ $\therefore$ only trivial solution.

$\therefore$ β is lin. ind $\therefore$ β is a basis

$\therefore$ $\dim(W) = 3$

$\dim(P_3) = 3+1 = 4$

$\dim(\{\underline{0}\}) = 0$

$(p(x))_\beta = (c_1, c_2, c_3) = (1, 1, -3)$.

$1+x+x^2-3x^3 = c_1(-1+x) + c_2(-1+x^2) + c_3(-1+x^3)$

$\therefore$ $c_3 = -3$, $c_1 = 1$, $c_2 = 1$

Question 4. *(3.5 marks)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

There are only three distinct two-dimensional subspaces of $\mathbb{P}_2$.

False

$$W_1 = \text{span}(\overbrace{\{1, x\}}^{B_1})$$
$$W_2 = \text{span}(\underbrace{\{1, x^2\}}_{B_2})$$
$$W_3 = \text{span}(\underbrace{\{x, x^2\}}_{B_3})$$
$$W_4 = \text{span}(\underbrace{\{1+x+x^2, 1+2x\}}_{B_4})$$

$\dim(W_i) = 2$ since B_i is a basis for W_i since the polynomials are not multiples of each other, therefore linearly independent.

$1+x \in W_1$ but $1+x \notin W_2, W_3, W_4$

$1+x^2 \in W_2$ but $1+x^2 \notin W_1, W_3, W_4$

$x+x^2 \in W_3$ but $x+x^2 \notin W_1, W_2, W_4$

$1+x+x^2 \in W_4$ but $1+x+x^2 \notin W_1, W_2, W_3$