Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S8: Winter 2025: **Quiz 2** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks each)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. If A and B are square matrices of the same size and c is a scalar then $\operatorname{tr}(A^T + cB) = \operatorname{tr}(A) + c\operatorname{tr}(B)$.

True, Let $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$

$$\begin{aligned}
LHS &= tr(A^T + cB) \\
&= tr([a_{ij}]^T + c[b_{ij}]) \\
&= tr([a_{ji}] + [cb_{ij}]) \\
&= tr([a_{ji} + cb_{ij}]) \\
&= (a_{11} + cb_{11}) + (a_{22} + cb_{22}) + \dots + (a_{nn} + c\, b_{nn}) \\
&= (a_{11} + a_{22} + \dots + a_{nn}) + c(b_{11} + b_{22} + \dots + b_{nn}) \\
&= tr(A) + c\,tr(B)
\end{aligned}$$

Question 3. *(5 marks)* Show that the reduced row echelon form of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if $ad - bc \neq 0$.

Case 1: $a \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim_{aR_2 \to R_2} \begin{bmatrix} a & b \\ ac & ad \end{bmatrix}$$

$$\sim_{-cR_1 + R_2 \to R_2} \begin{bmatrix} a & b \\ 0 & ad-bc \end{bmatrix}$$

$$\sim_{\frac{1}{ad-bc} R_2 \to R_2} \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \text{ since } ad-bc \neq 0$$

$$\sim_{-bR_2 + R_1 \to R_1} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sim_{\frac{1}{a}R_1 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ since } a \neq 0$$

Case 2: $a = 0$

$(0)d - bc \neq 0$

$bc \neq 0 \Rightarrow b \neq 0,\ c \neq 0$

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$$

$$\sim_{\frac{1}{b}R_2 \to R_2} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} \text{ since } b \neq 0$$

$$\sim_{-dR_2 + R_1 \to R_1} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sim_{\frac{1}{c}R_1 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ since } c \neq 0.$$

Question 2. *(3 marks)* Find all 2×2 matrices M such that $MA - AM = 0$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$0 = MA - AM$

$$0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$0 = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} - \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

$$0 = \begin{bmatrix} -c & a-d \\ 0 & c \end{bmatrix}$$

$\Rightarrow c = 0$

$b = s \in \mathbb{R}$

$d = t \in \mathbb{R}$

$\to a = d = t$

$\therefore (a,b,c,d) = (t, s, 0, t) \quad s, t \in \mathbb{R}$

Question 3. *(6 marks)* Find the values, if any, of k for which the following system has:

$$\begin{cases} x + y + kz = 1 \\ x + ky + z = 1 \\ kx + y + z = 1 \end{cases}$$

Exactly one solution, no solutions, infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 1 & K & 1 \\ 1 & K & 1 & 1 \\ K & 1 & 1 & 1 \end{array}\right]$$

$$\underset{\substack{-R_1+R_2\to R_2 \\ -KR_1+R_3\to R_3}}{\sim} \left[\begin{array}{cccc} 1 & 1 & K & 1 \\ 0 & K-1 & 1-K & 0 \\ 0 & 1-K & 1-K^2 & 1-K \end{array}\right]$$

$$\underset{R_2+R_3\to R_3}{\sim} \left[\begin{array}{cccc} 1 & 1 & K & 1 \\ 0 & K-1 & 1-K & 0 \\ 0 & 0 & 2-K-K^2 & 1-K \end{array}\right] = \left[\begin{array}{cccc} 1 & 1 & K & 1 \\ 0 & K-1 & 1-K & 0 \\ 0 & 0 & -(K-1)(K+2) & 1-K \end{array}\right]$$

If $K=1$ then $\left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

the system has ∞-many solutions since the #leading 1 < #var.

If $K \neq 1, -2$ then the system has a unique solution since #leading 1 = #var.

If $K=-2$ then $\left[\begin{array}{cccc} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$

the system has no solution since there is a leading entry in the constant column.

Bonus Question. *(3 marks)* If A, B and C are matrices such that the operations are defined, show that $A(BC) = (AB)C$.