Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S8: Winter 2025: **Quiz 3** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks each)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. If A is row equivalent to an elementary matrix, then the system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$.

True,

$A \sim$ K elementary row operation $\sim E \sim$ inverse elementary row operation from which E was obtained $\sim I$

$\therefore$ the RREF of A is I $\therefore$ by the equivalence theorem $A\underline{x} = \underline{b}$ is consistent $\forall \underline{b}$

Question 2. *(5 marks)* Solve for X where $\left(\text{trace}(2E)I + X^T\right)^{-1} = \frac{1}{3}\left(X^{-1}\right)^T E^T$ and $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = E^T = E^{-1}$

$$\left(\left(tr(2E)I + X^T\right)^{-1}\right)^{-1} = \left(\tfrac{1}{3}(X^{-1})^T E^T\right)^{-1}$$

$$tr(2E)I + X^T = 3\,(E^T)^{-1}\left((X^T)^{-1}\right)^{-1}$$

$$2I + X^T = 3EX^T$$

$$2I = 3EX^T - X^T$$

$$2I = \left(3E - I\right)X^T$$

$$2I = \underbrace{\begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}}_{B} X^T$$

$$B^{-1}(2I) = B^{-1}BX^T$$

$$2B^{-1} = X^T$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \begin{matrix} \tfrac{1}{2}R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} -1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & 8 & 3 & 0 & 1 \end{array}\right]$$

$$\sim \begin{matrix} -R_1 \to R_1 \\ \tfrac{1}{8}R_3 \to R_3 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & 1 & \tfrac{3}{8} & 0 & \tfrac{1}{8} \end{array}\right]$$

$$\sim 3R_3 + R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} \tfrac{1}{8} & 0 & \tfrac{3}{8} \\ 0 & \tfrac{1}{2} & 0 \\ \tfrac{3}{8} & 0 & \tfrac{1}{8} \end{matrix}}_{B^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$X^T = 2B^{-1}$$

$$(X^T)^T = (2B^{-1})^T$$

$$X = 2\begin{bmatrix} \tfrac{1}{8} & 0 & \tfrac{3}{8} \\ 0 & \tfrac{1}{2} & 0 \\ \tfrac{3}{8} & 0 & \tfrac{1}{8} \end{bmatrix}$$

$$X = \begin{bmatrix} \tfrac{1}{4} & 0 & \tfrac{3}{4} \\ 0 & 1 & 0 \\ \tfrac{3}{4} & 0 & \tfrac{1}{4} \end{bmatrix}$$

Question 3. *(3 marks)* Prove that if A and B are invertible matrices and $(AB)^2 = A^2B^2$ then $AB = BA$

premise:

- A and B are invertible, $AA^{-1} = I = A^{-1}A$
  $BB^{-1} = I = B^{-1}B$
- $(AB)^2 = A^2B^2$

Conclusion: $AB = BA$

From premise

$$(AB)^2 = A^2B^2$$

$$ABAB = AABB$$

$$A^{-1}ABABB^{-1} = A^{-1}AABBB^{-1}$$

$$IBAI = IABI$$

$$BA = AB$$

Question 4. *(5 marks)* Express

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \sim R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \overset{2R_3 + R_1 \to R_1}{\tfrac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since the RREF of A is I, A is invertible by equivalence theorem

and A^{-1} as a product of elementary matrices.

$$E_3E_2E_1A = I$$

$$E_3E_2E_1AA^{-1} = IA^{-1}$$

$$A^{-1} = E_3E_2E_1$$

$$I_3 \sim R_1 \leftrightarrow R_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$I_3 \sim 2R_3 + R_1 \to R_1 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

$$I_3 \sim \tfrac{1}{2}R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

$$I_3 \sim R_1 \leftrightarrow R_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1^{-1}$$

$$I_3 \sim -2R_3 + R_1 \to R_1 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2^{-1}$$

$$I_3 \sim 2R_2 \to R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3^{-1}$$

$$A^{-1} = E_3E_2E_1$$

$$(A^{-1})^{-1} = (E_3E_2E_1)^{-1}$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}$$