Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S8: Winter 2025: **Quiz 4** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks each)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

a. If A is any skew-symmetric matrix then A^2 is symmetric matrix.

True, premise: A is skew-symmetric, $A^T = -A$

Conclusion: $A^2 = A$ is symmetric, $(A^2)^T = A^2$

$$\begin{aligned} LHS &= (A^2)^T \\ &= (AA)^T \\ &= A^T A^T \\ &= -A(-A) \quad \text{by premise} \\ &= AA \\ &= A^2 \\ &= RHS \end{aligned}$$

b. If A^2 is a symmetric matrix, then A is a symmetric matrix.

False, Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = AA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is symmetric but A is not symmetric.

Question 2. *(5 marks)* Only using elementary operations show that

$$\begin{vmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} = 9 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

$$LHS = \underset{-2R_3+R_1 \to R_1}{=} \begin{vmatrix} p-4x & q-4y & r-4z \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$\underset{4R_2+R_1 \to R_1}{=} \begin{vmatrix} 9p & 9q & 9r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$\underset{\frac{1}{9}R_1 \to R_1}{=} 9 \begin{vmatrix} p & q & r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$\underset{-2R_1+R_2 \to R_2}{=} 9 \begin{vmatrix} p & q & r \\ x & y & z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$\underset{-2R_2+R_3 \to R_3}{=} 9 \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix}$$

$$\underset{R_1 \leftrightarrow R_3}{=} -9 \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$$

$$\underset{R_2 \leftrightarrow R_3}{=} 9 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

Question 3. *(5 marks)* If A is an $n \times n$ matrix, the *characteristic polynomial* $c_A(x)$ of A is defined by $c_A(x) = \det(xI - A)$.

a. *(5 marks)* Find the eigenvalues λ of $A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$. That is, find the values of λ for which $c_A(\lambda) = 0$.

b. *(3 marks bonus)* Find the eigenvector for a non-zero eigenvalue found in part a. That is, find a non-trivial solution $\mathbf{x}$ for each λ where $A\mathbf{x} = \lambda\mathbf{x}$.

$$0 = \det\left(\lambda I - \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}\right)$$

$$= \begin{vmatrix} \lambda-2 & -2 & 2 \\ -1 & \lambda-3 & 1 \\ 1 & -1 & \lambda-1 \end{vmatrix}$$

$$= \underset{C_3+C_2 \to C_2}{} \begin{vmatrix} \lambda-2 & 0 & 2 \\ -1 & \lambda-2 & 1 \\ 1 & \lambda-2 & \lambda-1 \end{vmatrix}$$

$$= \underset{R_2+R_3 \to R_3}{} \begin{vmatrix} \lambda-2 & 0 & 2 \\ -1 & \lambda-2 & 1 \\ 0 & 2\lambda-4 & \lambda \end{vmatrix}$$

$$= (\lambda-2)\begin{vmatrix} \lambda-2 & 1 \\ 2\lambda-4 & \lambda \end{vmatrix} + 2\begin{vmatrix} -1 & \lambda-2 \\ 0 & 2\lambda-4 \end{vmatrix}$$

$$= (\lambda-2)\left[(\lambda-2)\lambda - (2\lambda-4)\right] - 2\left[2\lambda-4\right]$$

$$= (\lambda-2)\left[\lambda^2 - 2\lambda - 2\lambda + 4\right] - 4\left[\lambda-2\right]$$

$$= (\lambda-2)\left[\lambda^2 - 4\lambda\right]$$

$$= (\lambda-2)\lambda(\lambda-4)$$

$\lambda = 2 \quad \lambda = 0 \quad \lambda = 4$

b) $A\underline{x} = \lambda\underline{x}$

$A\underline{x} - \lambda\underline{x} = \underline{0}$

$(A - \lambda I)\underline{x} = \underline{0}$

$\lambda = 2$: $\begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}$

$\sim_{R_1+R_3 \to R_3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix}$

$\sim_{\frac{1}{2}R_2 \to R_2,\ -R_2+R_3 \to R_3} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim_{-R_2+R_1 \to R_1} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\sim_{-R_1 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Let $z = t \quad t \in \mathbb{R}$

$x = 0$

$y = t$

$\therefore \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\lambda = 2$.