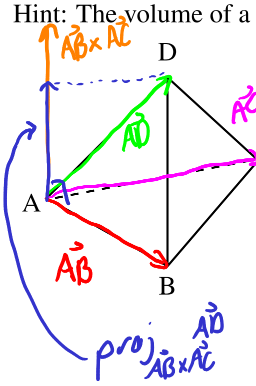
Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S8: Winter 2025: **Quiz 7** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(5 marks)* Show that the volume of the pyramid with vertices A, B, C, and D is $\frac{1}{6}|\vec{AB}\cdot(\vec{AC}\times\vec{AD})|$.
Hint: The volume of a pyramid with base area A and height h is $\frac{1}{3}Ah$.



$$\text{Volume} = \frac{1}{3}\text{Area of triangle} \times \text{height}$$

$$= \frac{1}{3}\frac{1}{2}\|\vec{AB}\times\vec{AC}\|\,\|\text{proj}_{\vec{AB}\times\vec{AC}}\vec{AD}\|$$

$$= \frac{1}{6}\|\vec{AB}\times\vec{AC}\|\left\|\frac{\vec{AD}\cdot(\vec{AB}\times\vec{AC})}{(\vec{AB}\times\vec{AC})\cdot(\vec{AB}\times\vec{AC})}\vec{AB}\times\vec{AC}\right\|$$

$$= \frac{1}{6}\cancel{\|\vec{AB}\times\vec{AC}\|}\frac{|\vec{AD}\cdot(\vec{AB}\times\vec{AC})|}{\cancel{\|\vec{AB}\times\vec{AC}\|^2}}\cancel{\|\vec{AB}\times\vec{AC}\|}$$

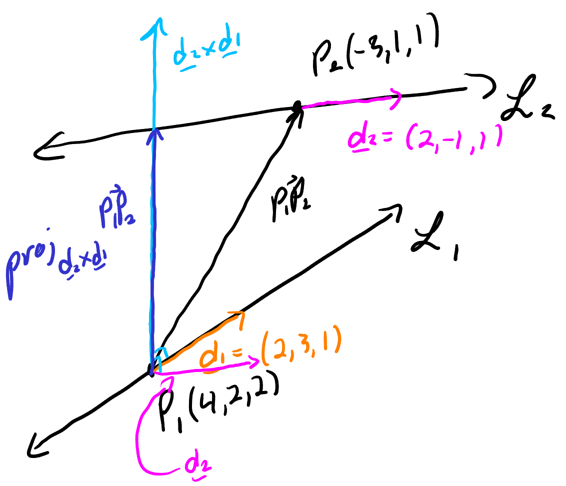
$$= \frac{1}{6}|\vec{AD}\cdot(\vec{AB}\times\vec{AC})|$$

$$= \frac{1}{6}\left|\begin{vmatrix} (\vec{AD})_1 & (\vec{AD})_2 & (\vec{AD})_3 \\ (\vec{AB})_1 & (\vec{AB})_2 & (\vec{AB})_3 \\ (\vec{AC})_1 & (\vec{AC})_2 & (\vec{AC})_3 \end{vmatrix}\right|$$

$$= \frac{1}{6} \overset{R_2 \leftrightarrow R_3}{} \left|\begin{vmatrix} (\vec{AD})_1 & (\vec{AD})_2 & (\vec{AD})_3 \\ (\vec{AC})_1 & (\vec{AC})_2 & (\vec{AC})_3 \\ (\vec{AB})_1 & (\vec{AB})_2 & (\vec{AB})_3 \end{vmatrix}\right|$$

$$\overset{R_1 \leftrightarrow R_3}{=} \frac{1}{6}\left|\begin{vmatrix} (\vec{AB})_1 & (\vec{AB})_2 & (\vec{AB})_3 \\ (\vec{AC})_1 & (\vec{AC})_2 & (\vec{AC})_3 \\ (\vec{AD})_1 & (\vec{AD})_2 & (\vec{AD})_3 \end{vmatrix}\right| = \frac{1}{6}|\vec{AB}\cdot(\vec{AC}\times\vec{AD})|$$

Question 2. *(5 marks)* Find the distance between the following skew lines $\mathscr{L}_1 : \begin{cases} x = 4+2t \\ y = 2+3t \\ z = 2+t \end{cases}$, and $\mathscr{L}_2 : \begin{cases} x = -3+2s \\ y = 1-s \\ z = 1+s \end{cases}$, $s, t \in \mathbb{R}$.



$$\text{distance} = \|\text{proj}_{\vec{d_2}\times\vec{d_1}}\vec{P_1P_2}\|$$

$$= \left\|\frac{(-4,0,8)\cdot(-7,-1,-1)}{(-4,0,8)\cdot(-4,0,8)}(-4,0,8)\right\|$$

$$= \left\|\frac{20}{80}(-4,0,8)\right\|$$

$$= \frac{1}{4}\sqrt{(-4)^2+0^2+8^2}$$

$$= \frac{1}{4}\sqrt{80}$$

$$= \frac{1}{4}\sqrt{16}\sqrt{5}$$

$$= \sqrt{5}$$

$$\vec{P_1P_2} = \vec{OP_2} - \vec{OP_1} = (-3,1,1) - (4,2,2) = (-7,-1,-1)$$

$$\begin{matrix} 2 & 2 \\ -1 & 3 \\ 1 & 1 \end{matrix} \qquad \vec{d_2}\times\vec{d_1} = \left(\begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 2 \\ -1 & 3 \end{vmatrix}\right) = (-4,0,8)$$

Question 3. Consider the set

$$V = \{(x,\ y) \mid x \geq 0 \text{ and } y \geq 0\}$$

under the following operations:

$$(x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 y_2) \qquad k(x,\ y) = (kx,\ y)$$

a. *(2 marks)* Does V contain a zero vector? If so find it. Justify.

b. *(2 marks)* Does V contain the additive inverse (negative of the vector in the sense of a vector space) of $\vec{v} = (3,\ 2)$? If so find it. Justify.

c. *(1 mark)* Is V a vector space? Justify.

a) Let $\underline{v} = (x,y) \in V$ and $\underline{0} = (a,b)$

$$\underline{v} + \underline{0} = \underline{v}$$
$$(x,y) + (a,b) = (x,y)$$
$$(x+a,\ yb) = (x,y)$$

$x + a = x \quad yb = y$

$a = 0 \quad \Rightarrow b = 1$

$\therefore \underline{0} = (0,1) \in V$

b) Let $\underline{w} = (x,y)$

$$\underline{v} + \underline{w} = \underline{0}$$
$$(3,2) + (x,y) = (0,1)$$
$$(3+x,\ 2y) = (0,1)$$

$3 + x = 0 \quad 2y = 1$

$x = -3 \quad y = \frac{1}{2}$

$\therefore \underline{w} = (-3, \frac{1}{2}) \notin V$ since $-3 \ngeq 0$.

c) Not a V.S. because not all vectors have additive inverses.

Question 4. *(5 marks)* Let $W = \{(f \mid f(-x) = f(x)\}$. Determine whether W is a subspace of $V = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$.

$W \neq \{\}$ since $0(x) = 0 \in W$ because $0(-x) = 0 = 0(x)$

① Closure under addition

Let $f, g \in W \Rightarrow f(-x) = f(x)$
$g(-x) = g(x)$

$f + g \in W$ since
$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$$

② Closure under scalar multiplication

Let $f \in W \Rightarrow f(-x) = f(x)$

$r \in \mathbb{R}$

$rf \in W$ since $(rf)(-x) = rf(-x) = r(f(x)) = (rf)(x)$

$\therefore W$ is a subspace by the subspace test.

Bonus. *(3 marks)* Sketch $r(t) = (\sin t,\ \cos t,\ t)$ where $t \in \mathbb{R}$.