Dawson College: **Linear Algebra** (SCIENCE): 201-NYC-05-S8: Winter 2025: **Quiz 8** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(5 marks)* Let V and W be subspaces of $\mathbb{R}^2$ that are spanned by $(3,\ 1)$ and $(2,\ 1)$, respectively. Find a vector $\mathbf{v}$ in V and a vector $\mathbf{w}$ in W for which $\mathbf{v}+\mathbf{w}=(3,\ 5)$.

$V = \text{span}(\{(3,1)\})$ $\qquad$ $\underline{v} \in V \Rightarrow \underline{v} = c_1(3,1)$

$W = \text{span}(\{(2,1)\})$ $\qquad$ $\underline{w} \in W \Rightarrow \underline{w} = k_1(2,1)$

$$\underline{v} + \underline{w} = (3,5)$$

$$c_1(3,1) + k_1(2,1) = (3,5)$$

$$\begin{bmatrix} 3 & 2 & 3 \\ 1 & 1 & 5 \end{bmatrix} \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 5 \\ 3 & 2 & 3 \end{bmatrix}$$

$$\sim_{-3R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & -12 \end{bmatrix}$$

$$\sim_{\substack{R_2 + R_1 \to R_1 \\ -R_2 \to R_2}} \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 12 \end{bmatrix}$$

$\therefore$ $\underline{v} = -7(3,1) = (-21,-7)$

$\underline{w} = 12(2,1) = (24, 12)$

Question 2. *(5 marks)* Prove that if $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ is linearly independent and $\mathbf{v}_3$ does not lie in span$\{\mathbf{v}_1,\ \mathbf{v}_2\}$, then $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ is linearly independent.

Suppose $S = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is linearly dependent. Then $\exists c_i \neq 0$ s.t. $\underline{0} = c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3$

<u>Case $c_3 \neq 0$:</u>

$$\underline{0} = c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3$$

$$-c_3\underline{v}_3 = c_1\underline{v}_1 + c_2\underline{v}_2$$

$$\underline{v}_3 = -\frac{c_1}{c_3}\underline{v}_1 - \frac{c_2}{c_3}\underline{v}_2 \Rightarrow \underline{v}_3 \in \text{span}(\{\underline{v}_1, \underline{v}_2\})$$ ↯ since $\underline{v}_3 \notin \text{span}(\{\underline{v}_1, \underline{v}_2\})$

<u>Case $c_3 = 0$:</u> $\Rightarrow c_1 \neq 0$ or $c_2 \neq 0$

$$\underline{0} = c_1\underline{v}_1 + c_2\underline{v}_2 + 0\underline{v}_3$$

$$\underline{0} = c_1\underline{v}_1 + c_2\underline{v}_2$$

$\therefore$ $\{\underline{v}_1, \underline{v}_2\}$ is linearly dependent because non-trivial linear combination that gives $\underline{0}$. ↯ since $\{\underline{v}_1, \underline{v}_2\}$ is linearly independent.

$\therefore$ S is linearly independent.

Question 3. Let W be the subspace of all polyomials in $\mathbb{P}_3$ such that $p(1) = 0$

a. *(4 marks)* Find a basis $\mathscr{B}$ of W.

b. *(1.5 marks)* State $\dim(\mathbb{P}_3)$, $\dim(W)$, and $\dim(\{0+0x+0x^2+0x^3\})$.

c. *(1 mark)* Find the coordinate vector of $p(x) = 1+x+x^2-3x^3$ relative to the basis found in part a.

Let $p(x) = a+bx+cx^2+dx^3 \in P_3$

$$0 = p(1)$$
$$0 = a+b+c+d$$
$$a = -b-c-d$$

$\therefore$ $p(x) = (-b-c-d) + bx + cx^2 + dx^3 \in W$

$$= b\underbrace{(-1+x)}_{p_1(x)} + c\underbrace{(-1+x^2)}_{p_2(x)} + d\underbrace{(-1+x^3)}_{p_3(x)}$$

$\therefore$ $\beta = \{p_1(x), p_2(x), p_3(x)\}$ spans W

<u>Linear independence:</u>

$$\underline{0} = c_1(-1+x) + c_2(-1+x^2) + c_3(-1+x^3)$$
$$\underline{0} = (-c_1-c_2-c_3)(1) + c_1(x) + c_2(x^2) + c_3(x^3)$$

$\Rightarrow c_1 = c_2 = c_3 = 0$ $\therefore$ only trivial solution.

$\therefore$ β is lin. ind $\therefore$ β is a basis

$\therefore$ $\dim(W) = 3$

$\dim(P_3) = 3+1 = 4$

$\dim(\{0\}) = 0$

$(p(x))_\beta = (c_1, c_2, c_3) = (1, 1, -3)$.

$1+x+x^2-3x^3 = c_1(-1+x) + c_2(-1+x^2) + c_3(-1+x^3)$

$\therefore$ $c_3 = -3$, $c_1 = 1$, $c_2 = 1$

Question 4. *(3.5 marks)* Determine whether the following statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

Every basis of $\mathbb{P}_4$ contains at least one polynomial of degree 3 or less.

False

$\beta = \{1+x^4, x+x^4, x^2+x^4, x^3+x^4, x^4\}$ is a basis of P_4.

β is linearly independent: $0 = c_1(1+x^4) + c_2(x+x^4) + c_3(x^2+x^4) + c_4(x^3+x^4) + c_5 x^4$

$$0 = c_1(1) + c_2(x) + c_3(x^2) + c_4(x^3) + (c_1+c_2+c_3+c_4+c_5)x^4$$

$c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_1+c_2+c_3+c_4+c_5 = 0$

$c_5 = 0$

Only trivial solution, $\therefore$ β lin. ind.

β spans P_4 since $\dim(P_4) = 4+1 = 5 = \#$vectors in β and β is lin. ind.

$\therefore$ β is a basis.

Bonus. *(1 mark)* State your favorite proof in Linear Algebra.