Dawson College: **Integral Calculus**: 201-SN3-RE-S2: Winter 2026: **Quiz 7** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(5 marks each)* Determine whether each series is absolutely convergent, conditionally convergent, or divergent. Justify your answer carefully, and clearly state the test that you use.

a. $$\sum_{n=1}^{\infty}(-1)^n\frac{3^n n!}{n^n}$$

Lets apply the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}}{(-1)^n\frac{3^n n!}{n^n}}\right| = \lim_{n\to\infty}\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}\frac{n^n}{3^n n!}$$

$$= \lim_{n\to\infty}\frac{\cancel{3^n}3\,\cancel{n!}\,\cancel{(n+1)}\,n^n}{(n+1)^n\cancel{(n+1)}\cancel{3^n}\cancel{n!}}$$

$$= 3\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \quad \text{I.F. } 1^\infty$$

$$y = \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x$$

$$\ln y = \ln \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x$$

$$\ln y = \lim_{x\to\infty}\ln\left(\frac{x}{x+1}\right)^x$$

$$\ln y = \lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)$$

$$\ln y = \lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}}$$

$$\ln y \overset{H}{=} \lim_{x\to\infty}\frac{\frac{1}{\frac{x}{x+1}}\cdot\frac{1(x+1)-x}{(x+1)^2}}{\frac{-1}{x^2}}$$

$$\ln y = \lim_{x\to\infty}\frac{\cancel{x+1}}{x}\cdot\frac{1}{(x+1)^{\cancel{2}}}\cdot -x^{\cancel{2}}$$

$$\ln y = -1$$

$$y = e^{-1}$$

$$= 3e^{-1} > 1$$

$\therefore$ diverges by ratio test

b. $$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$$

Lets apply the integral test.

Let $f(x) = \dfrac{1}{x(\ln x)^3}$

- $f(x)$ is continuous on $[2,\infty)$
- $f(x)$ is positive on $[2,\infty)$
- $f'(x) = \dfrac{-1}{(x(\ln x)^3)^2}\cdot\left((\ln x)^3 + \dfrac{x\,3(\ln x)^2}{x}\right) < 0$

$\therefore f(x)$ is decreasing on $[2,\infty)$

$$\int_2^\infty \frac{1}{x(\ln x)^3}dx = \lim_{b\to\infty}\int_2^b\frac{1}{x(\ln x)^3}dx$$

$u = \ln x$

$du = \frac{1}{x}dx$

$u(b) = \ln b$

$u(2) = \ln 2$

$$= \lim_{b\to\infty}\int_{\ln 2}^{\ln b}\frac{1}{u^3}du$$

$$= \lim_{b\to\infty}\left[\frac{-u^{-2}}{2}\right]_{\ln 2}^{\ln b}$$

$$= \lim_{b\to\infty}\left[\underbrace{\frac{-1}{2(\ln b)^2}}_{\to 0} + \frac{1}{2(\ln 2)^2}\right]$$

$$= \frac{1}{2(\ln 2)^2}$$

By the integral test since $\int_2^\infty \frac{1}{x(\ln x)^3}dx$ converges then so does $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^3}$

$\therefore$ absolutely convergent.

Question 2. *(5 marks each)* Determine whether each series is absolutely convergent, conditionally convergent, or divergent. Justify your answer carefully, and clearly state the test that you use.

a.

$$\underbrace{\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n}}_{a_n}$$

By inspection the series behaves like $\sum b_n$ where $b_n = \frac{1}{n^2}$.

$\therefore$ should converge since p-series where $p = 2 > 1$.

Lets apply the limit comparison test

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sin(\frac{1}{n})}{n}}{\frac{1}{n^2}} = \lim_{h\to\infty} \frac{n^{\cancel{2}}\sin(\frac{1}{n})}{\cancel{n}} \quad \text{I.F. } \infty \cdot 0$$

$$= \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \quad \text{I.F. } \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}}$$

$$\stackrel{\hat{H}}{=} \lim_{x\to\infty} \frac{\cos(\frac{1}{x})\,\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = 1 > 0 \text{ and finite}$$

$\therefore$ behaves like $\sum b_n$ $\therefore$ converges by limit comparison test.

$\therefore$ absolutely convergent

b.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \underbrace{\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}}}_{b_n}$$

Is the series absolutely convergent?

$$\sum_{n=1}^{\infty} \left|(-1)^{n-1} b_n\right| = \sum \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \sum \frac{n+1-n}{n+1+\sqrt{n}\sqrt{n+1}}$$

$$= \sum \frac{1}{n+1+\sqrt{n}\sqrt{n+1}}$$

$$b_n = \frac{1}{n+1+\sqrt{n}\sqrt{n+1}} \geq \frac{1}{n+n+\sqrt{n}\sqrt{n+1}}$$

$$\geq \frac{1}{2n+\sqrt{n}\sqrt{n+3n}} = \frac{1}{2n+\sqrt{n^2+3n^2}} = \frac{1}{2n+\sqrt{4n^2}} = \frac{1}{2n+2n} = \frac{1}{4n} = c_n$$

$\sum c_n$ is a p-series where $p = 1$ $\therefore$ diverges $\therefore$ $\sum b_n$ diverges by Comparison Test. $\therefore$ not absolutely convergent.

Lets apply the Alternating Series Test

1) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+1+\sqrt{n}\sqrt{n+1}} = 0$

2) $b_{n+1} \stackrel{?}{\leq} b_n$, let $f(x) = \frac{1}{x+1+\sqrt{x}\sqrt{x+1}}$, $f'(x) = \frac{-1}{(x+1+\sqrt{x}\sqrt{x+1})}\left(1 + \frac{\sqrt{x+1}}{2\sqrt{x}} + \frac{\sqrt{x}}{2\sqrt{x+1}}\right) < 0$

$\therefore$ $b_{n+1} < b_n$

$\therefore$ $\sum (-1)^{n-1} b_n$ convergent by Alternating Series Test

$\therefore$ $\sum (-1)^{n-1} b_n$ is conditionally convergent.