Dawson College: **Linear Algebra and Vector Geometry**: 201-SN4-RE-S6: Winter 2026: **Quiz 4** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. *(3 marks each)* Determine whether the following statements are true or false. If a statement is false provide a counterexample. If it is true provide a proof.

a. If A is row equivalent to I_n, and E is an elementary matrix, then the system $EA^2A^T\mathbf{x} = \mathbf{b}$ is consistent for any $n \times 1$ column matrix $\mathbf{b}$. True

A is invertible by the Equivalence thm. since its RREF is I. And it follows from thm. seen in class that A^2 and A^T are invertible. E is invertible since its an elem. mat. The product of invertible matrices is invertible. $\therefore$ EA^2A^T is invertible. So by the equivalence thm. $EA^2A^T\underline{x} = \underline{b}$ is consistent $\forall \underline{b}$.

b. If $\mathbf{x}_1$ is any solution to the linear system $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_0$ is any solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x}_k = \mathbf{x}_1 + k\mathbf{x}_0$ for any scalar k is a solution to the linear system $A\mathbf{x} = \mathbf{b}$.

True, From the premise we have $A\underline{x}_1 = \underline{b}$ and $A\underline{x}_0 = \underline{0}$. And since

$$\begin{aligned} A\underline{x}_k &= A(\underline{x}_1 + k\underline{x}_0) \\ &= A\underline{x}_1 + Ak\underline{x}_0 \\ &= \underline{b} + kA\underline{x}_0 \\ &= \underline{b} + k\underline{0} \\ &= \underline{b} \end{aligned}$$

$\therefore$ $\underline{x}_k$ is a solution.

Question 2. *(3 marks)* Prove: If $A^TA = A$, then A is symmetric and $A = A^2$.

premise: $A = A^TA$

conclusion: $A^T = A$

$A^2 = A$

$$\begin{aligned} A^T &= (A^TA)^T && \text{by premise} \\ &= A^T(A^T)^T \\ &= A^TA \\ &= A && \text{by premise} \end{aligned}$$

$$\begin{aligned} A^2 &= AA \\ &= A^TA && \text{since } A^T = A \\ &= A && \text{by premise} \end{aligned}$$

Question 3. *(3 marks)* An $n \times n$ matrix P is called an *idempotent matrix* if $P^2 = P$. Show using a proof by contradiction that I is the only invertible idempotent matrix.

Suppose I is not the only invertible idempotent matrix.

That is $\exists P \neq I$ s.t. $P^2 = P$

$$PP = P$$

$$P^{-1}PP = P^{-1}P \quad \text{since } P \text{ is invertible}$$

$$IP = I$$

$$P = I \quad \lightning$$

$\therefore$ I is the only invertible idempotent matrix

Question 4. *(3 marks)* Find all 2×2 skew symmetric matrices A such that $A^2 = -I$.

Let $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = \begin{bmatrix} -a^2 & 0 \\ 0 & -a^2 \end{bmatrix} = -a^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ $a^2 = 1 \Rightarrow a = \pm 1$

$\therefore$ $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$