Dawson College: **Linear Algebra and Vector Geometry**: 201-SN4-RE-S6: Winter 2026: **Quiz 8** name: Y. Lamontagne

Books, watches, notes or cell phones are **not** allowed. The **only** calculators allowed are the Sharp EL-531**. You **must** show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1.

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid x+2y-z=0 \text{ and } 3x+5y+z=0\}.$$

a. *(1 mark)* Describe geometrically the set W inside $\mathbb{R}^3$.
b. *(3 marks)* Find a basis for W.
c. *(1 mark)* State $\dim(W)$.

a) The set of points along the intersection of the two planes.

b) $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 1 & 0 \end{bmatrix}$

$\sim_{-3R_1+R_2\to R_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix}$

$\sim_{\substack{2R_2+R_1\to R_1 \\ -R_2\to R_2}} \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix}$

Let $z=t$ $t\in\mathbb{R}$

$x=-7t$
$y=4t$

$(x,y,z)=t\overbrace{(-7,4,1)}^{\underline{d}}$ $t\in\mathbb{R}$

$\beta=\{\underline{d}\}$ is the basis since it spans the line and it is linearly independent since it contains one vector and it is nonzero.

c) $\dim(W)=1$

Question 2. *(5 marks)* Let

$$\mathbf{v}_1=(1,2,-1), \qquad \mathbf{v}_2=(2,3,1), \qquad \mathbf{w}=(a,b,c)\in\mathbb{R}^3.$$

Find necessary and sufficient conditions on a,b,c so that $\mathbf{w}\in\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}$. When the condition is satisfied, express $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

So $\underline{w}\in \operatorname{span}\{\underline{v}_1,\underline{v}_2\}$ iff $\exists c_1,c_2$ s.t. $\underline{w}=c_1\underline{v}_1+c_2\underline{v}_2$

$\begin{bmatrix} 1 & 2 & a \\ 2 & 3 & b \\ -1 & 1 & c \end{bmatrix} \sim_{\substack{-2R_1+R_2\to R_2 \\ R_1+R_3\to R_3}} \begin{bmatrix} 1 & 2 & a \\ 0 & -1 & b-2a \\ 0 & 3 & a+c \end{bmatrix} \sim_{3R_2+R_3\to R_3} \begin{bmatrix} 1 & 2 & a \\ 0 & -1 & b-2a \\ 0 & 0 & c+3b-5a \end{bmatrix}$

For the system to be consisent and consequently for c_1 and c_2 to exists $c+3b-5a=0$
$c=5a-3b$

$\therefore$ $c=5a-3b$ iff $(a,b,c)\in\operatorname{span}\{\underline{v}_1,\underline{v}_2\}$

Question 3. *(5 marks)* Consider the set

$$S=\{(1,2,0),\ (2,5,1),\ (3,7,1)\}\subset\mathbb{R}^3.$$

a. *(3 marks)* Determine whether S is linearly independent.
b. *(1 mark)* Determine whether S spans $\mathbb{R}^3$.
c. *(1 mark)* Is S a basis for $\mathbb{R}^3$?

a) $c_1(1,2,0)+c_2(2,5,1)+c_3(3,7,1)=(0,0,0)$

$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 0 & 1 & 1 \end{bmatrix}}_{A}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$|A|=a_{31}C_{31}+a_{32}C_{32}+a_{33}C_{33}$
$=(-1)\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}+\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}$
$=-1(1(7)-2(3))+((1)(5)-2(2))$
$=-1+1$
$=0$

$\therefore$ by the equivalence thm, not only the trivial sol.
$\therefore$ linearly dependent.

b) $\dim(\mathbb{R}^3)=3$, in order to span $\mathbb{R}^3$ we need 3 linearly independent vectors. $\therefore$ S does not span $\mathbb{R}^3$

c) S is not a basis since it does not span.

Question 4. Let

$$B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \{(1,0,1),\ (1,1,0),\ (0,1,1)\}.$$

a. *(3 marks)* Prove that B is a basis for $\mathbb{R}^3$.

b. *(2 marks)* Find the coordinate vector $[\mathbf{x}]_B$ of $\mathbf{x} = (4,5,6)$.

c. *(2 mark)* Find the vector $\mathbf{y} \in \mathbb{R}^3$ such that

$$[\mathbf{y}]_B = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

a) Lets determine whethe B is lin. ind

$$\underline{0} = c_1 \underline{b_1} + c_2 \underline{b_2} + c_3 \underline{b_3}$$

$$(0,0,0) = c_1(1,0,1) + c_2(1,1,0) + c_3(0,1,1)$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \overset{0}{\widehat{a_{13}C_{13}}}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1 - (-1) = 2 \neq 0$$

$\therefore$ by equivalence thm, only the trivial sol.

$\therefore$ linearly independent.

Since $\dim(\mathbb{R}^3) = 3$ and B has 3 lin. ind. vectors then it spans $\mathbb{R}^3$

$\therefore$ B is a basis.

b)

$$(4,5,6) = c_1 \underline{b_1} + c_2 \underline{b_2} + c_3 \underline{b_3}$$

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 1 & 5 \\ 1 & 0 & 1 & 6 \end{bmatrix} \underset{-R_1+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & -1 & 1 & 2 \end{bmatrix}$$

$$\underset{R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 2 & 7 \end{bmatrix}$$

$$\underset{\substack{-R_2+R_1 \to R_1 \\ \frac{1}{2}R_3 \to R_3}}{\sim} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & \frac{7}{2} \end{bmatrix}$$

$$\underset{\substack{R_3+R_1 \to R_1 \\ -R_3+R_2 \to R_2}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 7/2 \end{bmatrix}$$

$$[x]_B = \left(\tfrac{5}{2}, \tfrac{3}{2}, \tfrac{7}{2}\right)$$

c)

$$\underline{y} = 2b_1 - \underline{b_2} + 3b_3$$

$$= 2(1,0,1) - (1,1,0) + 3(0,1,1)$$

$$= (1,2,5)$$

Question 5. Let

$$\mathbf{u}_1 = (1,1,0), \quad \mathbf{u}_2 = (2,3,1), \quad \mathbf{u}_3 = (0,1,1), \quad \mathbf{u}_4 = (3,4,1).$$

a. *(4 marks)* Determine which vector(s), if any, are linear combinations of the preceding ones.

b. *(2 marks)* Extract a basis from the list $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

c. *(1 mark)* State the dimension of the set of all linear combinations of these vectors.

Lets determine which vectors make the set linearly dep.

$$\underline{0} = c_1\underline{u_1} + c_2\underline{u_2} + c_3\underline{u_3} + c_4\underline{u_4}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 3 & 1 & 4 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\underset{-R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\underset{-R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{-2R_2+R_1 \to R_1}{\sim} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $u_3 = -2u_1 + u_2$

$u_4 = u_1 + u_2$

b) by removing $\underline{u_3}$ and $\underline{u_4}$ the set becomes lin. independent. and therefore a basis of the span.

c) 2 since the basis has 2 vectors.

Question 6. *(3 marks)* A student claims: *"If a set of three vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$, then each vector in the set must be nonzero, and no vector can be a linear combination of the other two."* Is the student correct? Give a careful justification.

True because the dim. of the space is 3 and the set of 3 vectors span the space then the set is linearly indepent.

$\therefore$ does not contain the zero vector and none is a lin. comb. of the other.

Bonus Question. *(3 marks)* Prove that no vector subspace can have exactly two vectors